

# New effect in wave-packet scattering of quantum fields

Kenji Nishiwaki

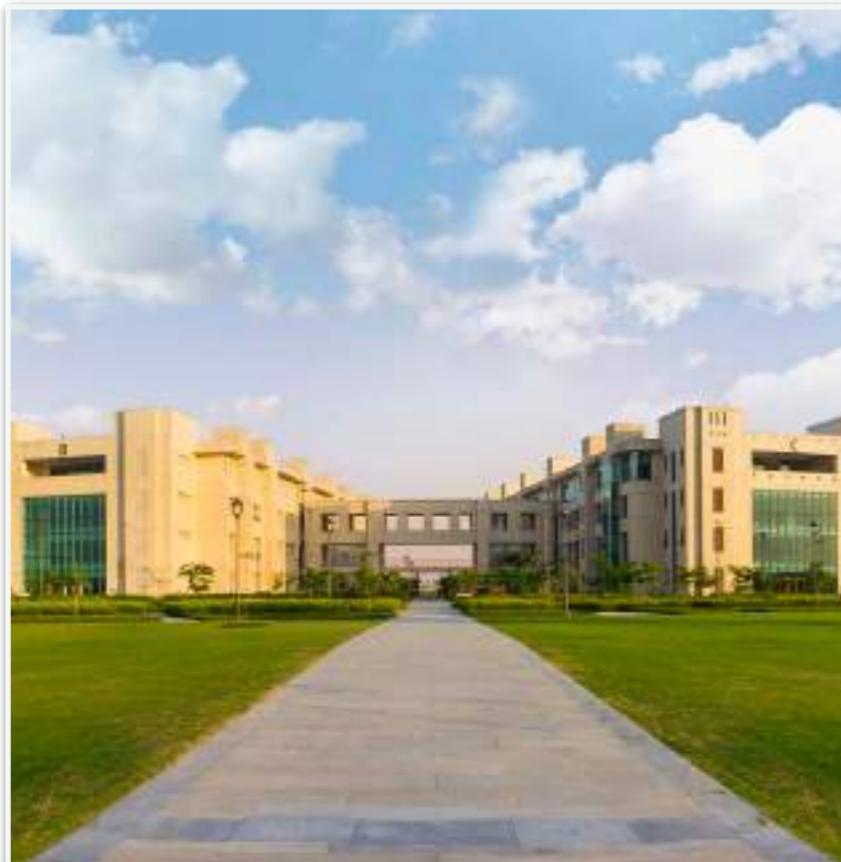
(केंजी निशिवाकि ← 니시와키 겐지 ← 西脇 健二)

**SHIV NADAR**  
UNIVERSITY  
DELHI NCR

Based on works with

Kenzo Ishikawa (Hokkaido) and Kin-ya Oda (Tokyo Woman's Christian)

[arXiv:2006.14159, 2102.12032 + ongoing]



# Intro: S-matrix in plane-wave basis

[QFT textbooks]

✓ The **standard tool** for describing quantum processes of **particles**:



Basis (@ Schrödinger Pic.):  $e^{i \mathbf{p} \cdot \mathbf{x}}$

(plane wave: the eigenstate of  $\mathbf{p}$ )  $\leftrightarrow$   $\mathbf{x}$  completely undetermined  
(non-normalisable mode)



$|S|^2$  is ill-defined due to  $|\delta^4(\mathbf{P}_{\text{out}} - \mathbf{P}_{\text{in}})|^2 = \delta^4(\mathbf{P}_{\text{out}} - \mathbf{P}_{\text{in}}) \times \underline{\delta^4(0)}$ .

$\Rightarrow$  Only the averaged (per V and T) frequencies of events is calculable.

[(Volume)(Time)  $\rightarrow \infty$ ]

  
 $\delta^4(0)$   
 decay widths & cross sections

$(T_{\text{in}} (= T_{\text{initial}}) = -\infty, T_{\text{out}} (= T_{\text{final}}) = +\infty)$

# Intro: Gaussian basis

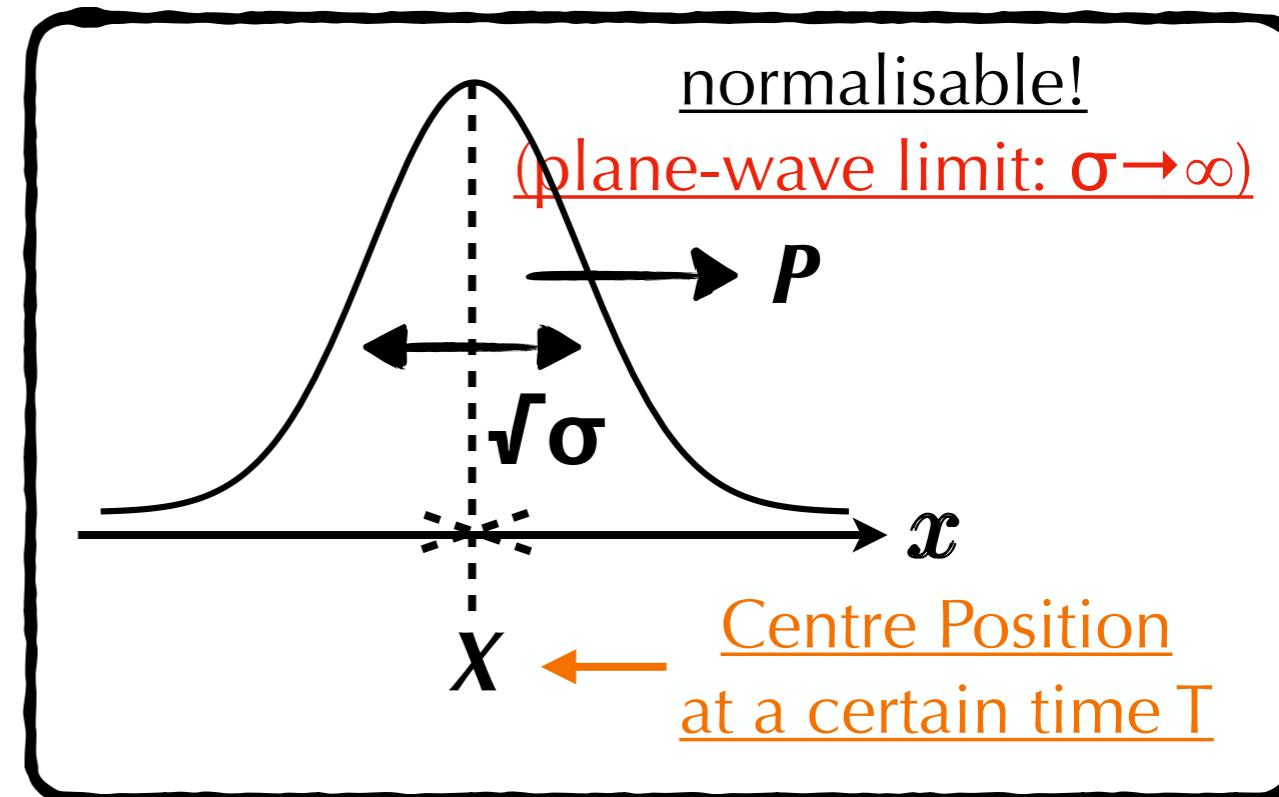
[Ishikawa, Shimomura (0508303), Ishikawa, Oda (1809.04285)]

- Key: Fields can be expanded in any complete sets of bases.  
→ Perturbations under **normalised** bases are possible. → **Gaussian!**

- Gaussian basis

- 📌 Form (@ Schrödinger Pic.):

$$\approx e^{i\mathbf{P} \cdot (\mathbf{x} - \mathbf{X}) - \frac{(\mathbf{x} - \mathbf{X})^2}{2\sigma}} \quad \begin{matrix} \text{(a coherent state)} & \text{(when } T=0) \end{matrix}$$



- 📌 Expansion of Scalar operator (in Int. Pic.):

$$\circ \hat{\phi}(x) = \int \frac{d^3 X d^3 P}{(2\pi)^3} [f_{\sigma, X, P}(x) \hat{A}(\sigma, X, P) + h.c.]$$

Wave function of Gaussian wave packet  
(X is defined @ T)

for the corresponding wave-packet state

$$\circ |\mathcal{P}\rangle = \hat{A}^\dagger(\mathcal{P}) |0\rangle, \quad \left[ \mathcal{P} = \underbrace{\{\sigma, X^0 (= T), X, P\}}_{=: X} \right]$$

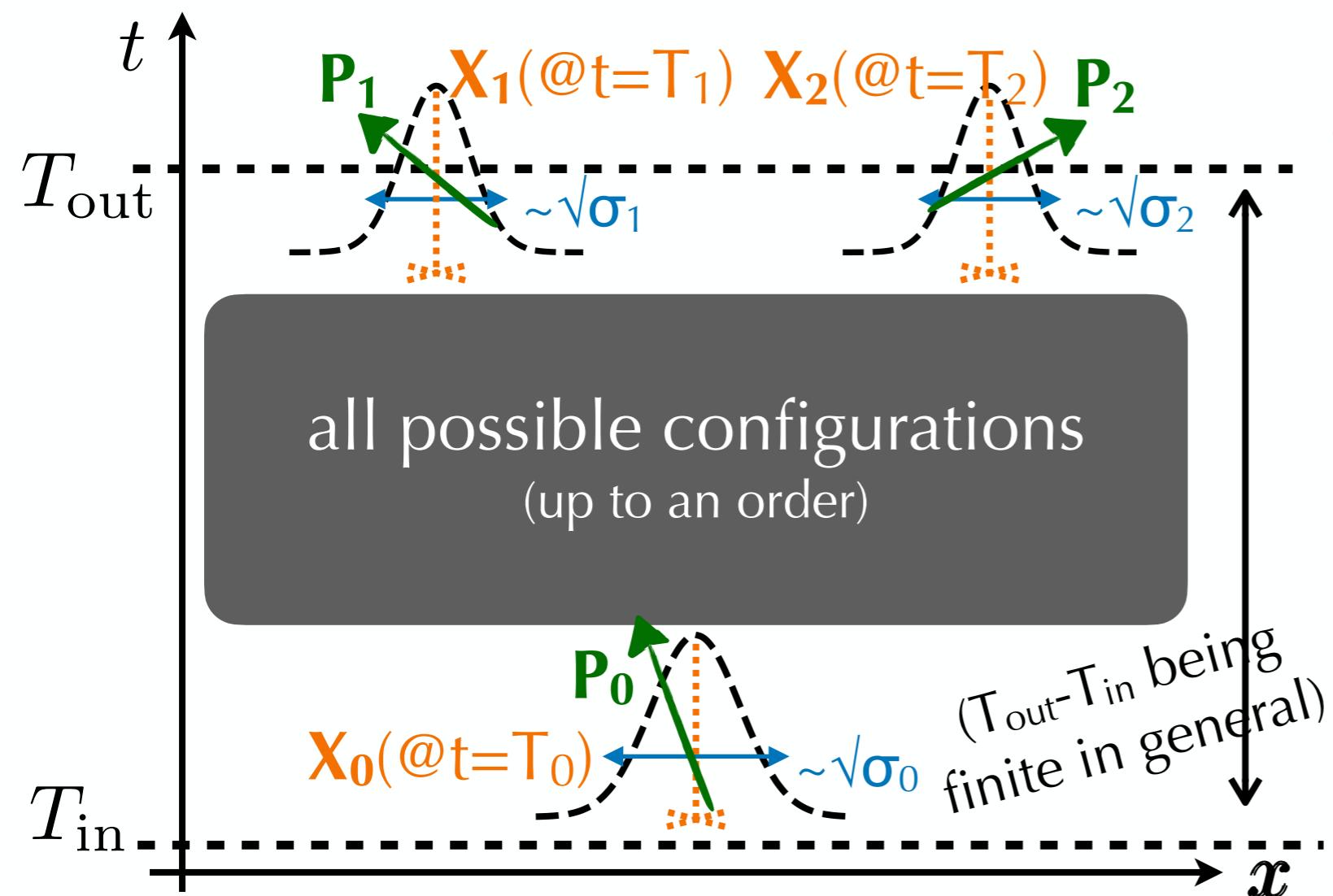
# Intro: S-matrix in Gaussian basis

## S-matrix ( $1 \rightarrow 2$ ) def.:

[Note: as in the plane-wave basis,  
but by the creation/annihilation  
operators for wave packets!]

$$\mathcal{S} := \langle \mathcal{P}_1, \mathcal{P}_2 | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} | \mathcal{P}_0 \rangle^{\text{free state}}_{\text{in}} \\ [\mathcal{P}_i = \{\sigma_i, \underbrace{X_i^0 (= T_i), X_i, P_i}_{=: X_i}\}]$$

This describes the amplitude for the **finite probability/frequency**  
of the **event with fully-described initial & final particle states!**



**Normalisability of Gaussian  
can makes  $S$  finite!**

- First proposal by coherent state:  
[Ishikawa, Shimomura (0508303)]
  - Claims on various phenomena  
by Ishikawa-san et. al.  
e.g. [Ishikawa, Jinnouchi, Kubota,  
Sloan, Tatsuishi (1901.03019)]
- Experiment by the same group → (1907.01264)

# Contents

1. (Intro.) S-matrix in Gaussian Wave Packet
- NEXT** 2. Properties of S-matrix for “ $1 \rightarrow 2$ ” [Ishikawa, Oda (1809.04285)]
3. Properties of S-matrix for “ $2 \rightarrow 2$ ” [Ishiwaka, KN, Oda  
(2006.14159, 2102.12032  
+ongoing)]

# S-matrix of the simplest $1 \rightarrow 2$ : $\Phi \rightarrow \phi\phi$

[Ishikawa, Oda (1809.04285)]

- When  $\hat{H}_{\text{int}}(t) = \int d^3x \frac{\kappa}{2} (\hat{\Phi} \hat{\phi} \hat{\phi})$ , for finite  $T_{\text{in}}$  &  $T_{\text{out}}$ ,  $S$  becomes

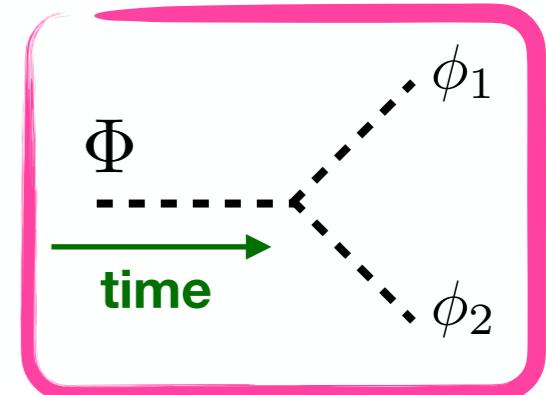
$$S := \langle \mathcal{P}_1, \mathcal{P}_2 | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} | \mathcal{P}_0 \rangle^{\text{free out-state}} \quad (\Pi_i := \{X_i, P_i\})$$

$$(\Pi_i := \{X_i, P_i\})$$



Wick's theorem  
for  $A$  and  $A^\dagger$  (@LO)

$$-\frac{i\kappa}{\sqrt{2}} \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3x f_{\phi, \sigma_1; \Pi_1}^*(x) f_{\phi, \sigma_2; \Pi_2}^*(x) f_{\Phi, \sigma_0; \Pi_0}(x)$$



# S-matrix of the simplest $1 \rightarrow 2$ : $\Phi \rightarrow \phi\phi$

[Ishikawa, Oda (1809.04285)]

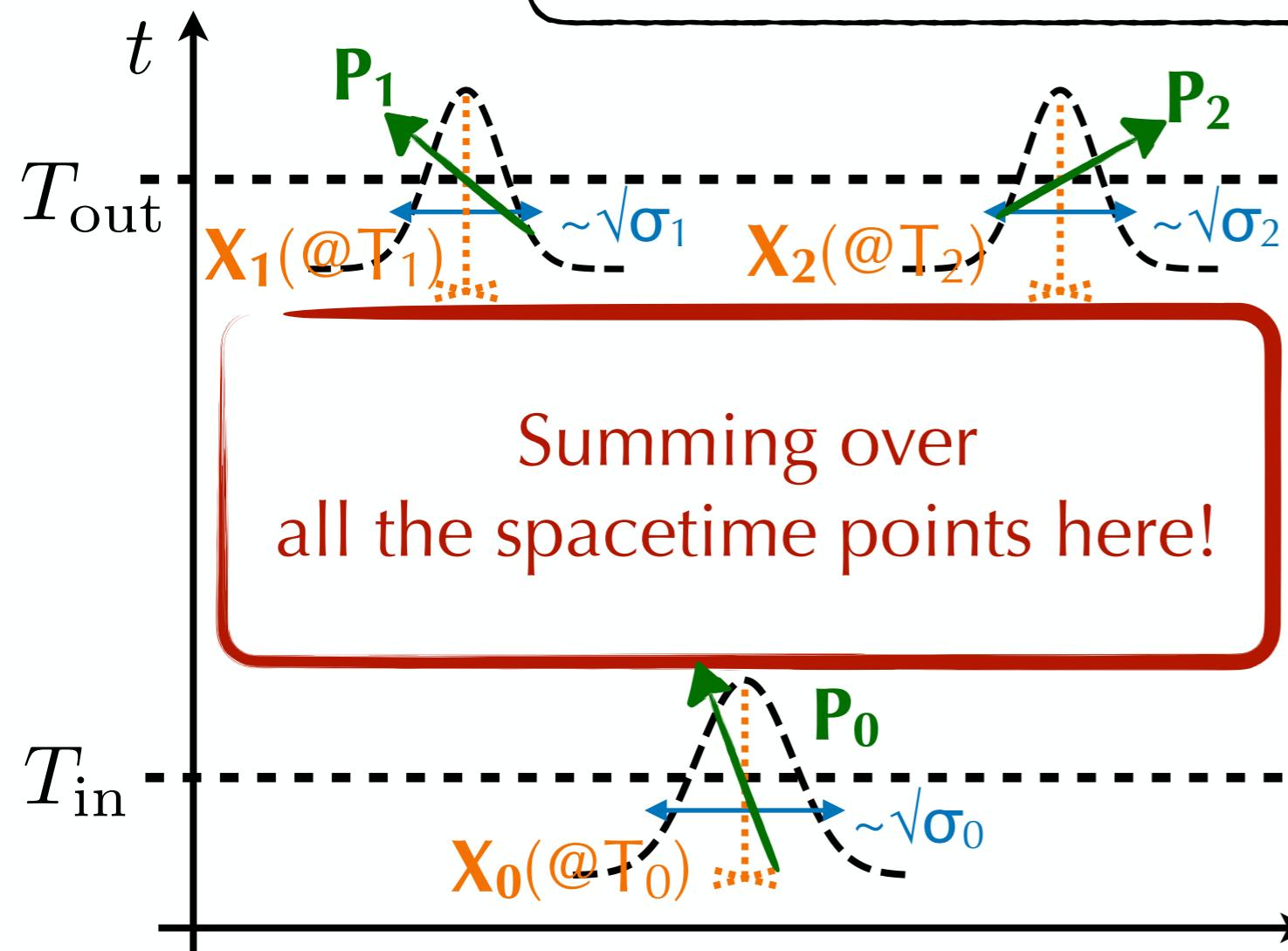
- When  $\hat{H}_{\text{int}}(t) = \int d^3x \frac{\kappa}{2} (\hat{\Phi}\hat{\phi}\hat{\phi})$ , for finite  $T_{\text{in}}$  &  $T_{\text{out}}$ ,  $S$  becomes

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“Wave-packet Feynman Rule”

# S-matrix of the simplest 1→2: $\Phi \rightarrow \phi\phi$

[Ishikawa, Oda (1809.04285)]

When  $\hat{H}_{\text{int}}(t) = \int d^3x \frac{\kappa}{2} (\hat{\Phi}\hat{\phi}\hat{\phi})$ , for finite  $T_{\text{in}}$  &  $T_{\text{out}}$ ,  $S$  becomes

$$S := \langle \mathcal{P}_1, \mathcal{P}_2 | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} | \mathcal{P}_0 \rangle$$

( $\Pi_i := \{X_i, P_i\}$ )

Wick's theorem  
for  $A$  and  $A^\dagger$  (@LO) →

$$-\frac{i\kappa}{\sqrt{2}} \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3x f_{\phi,\sigma_1;\Pi_1}^*(x) f_{\phi,\sigma_2;\Pi_2}^*(x) f_{\Phi,\sigma_0;\Pi_0}(x)$$

[Details (skippable)]

$$f_{\Psi,\sigma;\Pi}(x) = \left(\frac{\sigma}{\pi}\right)^{3/4} \int \frac{d^3p}{\sqrt{2p^0} (2\pi)^{3/2}} e^{ip \cdot (x - X) - \frac{\sigma}{2}(p - P)^2} \Bigg|_{p^0 = E_\Psi(p)}$$

saddle-point approx. for a large  $\sigma$

$$\left(\frac{\sigma}{\pi}\right)^{3/4} \left(\frac{2\pi}{\sigma}\right)^{3/2} \frac{1}{\sqrt{2P^0} (2\pi)^{3/2}} e^{iP \cdot (x - X) - \frac{(x - \Xi(t))^2}{2\sigma}} \Bigg|_{P^0 = E_\Psi(P)}$$

# S-matrix of the simplest $1 \rightarrow 2$ : $\Phi \rightarrow \phi\phi$

[Ishikawa, Oda (1809.04285)]

- When  $\hat{H}_{\text{int}}(t) = \int d^3x \frac{\kappa}{2} (\hat{\Phi}\hat{\phi}\hat{\phi})$ , for finite  $T_{\text{in}}$  &  $T_{\text{out}}$ ,  $S$  becomes

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Wick's theorem  
for  $A$  and  $A^\dagger$  (@LO)  $\longrightarrow$

$$-\frac{i\kappa}{\sqrt{2}} \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3x f_{\phi,\sigma_1;\Pi_1}^*(x) f_{\phi,\sigma_2;\Pi_2}^*(x) f_{\Phi,\sigma_0;\Pi_0}(x)$$

$$\Xi(t) := X + V_\Psi(\mathbf{P})(t - T)$$

$$f_{\Psi,\sigma;\Pi}(x) \simeq$$

Uniform linear motion  
of the centre (= Peak!)

$$\left(\frac{\sigma}{\pi}\right)^{3/4} \left(\frac{2\pi}{\sigma}\right)^{3/2}$$

$$\frac{1}{\sqrt{2P^0} (2\pi)^{3/2}} e^{i\mathbf{P} \cdot (\mathbf{x} - \mathbf{X}) - \frac{(x - \Xi(t))^2}{2\sigma}}$$

$$V_\Psi(\mathbf{P}) := \mathbf{P}/E_\Psi(\mathbf{P})$$

$$E_\Psi(\mathbf{P}) := \sqrt{\mathbf{P}^2 + m_\psi^2}$$

$$P^0 = E_\Psi(\mathbf{P})$$

# S-matrix of the simplest $1 \rightarrow 2$ : $\Phi \rightarrow \phi\phi$

[Ishikawa, Oda (1809.04285)]

- When  $\hat{H}_{\text{int}}(t) = \int d^3x \frac{\kappa}{2} (\hat{\Phi}\hat{\phi}\hat{\phi})$ , for finite  $T_{\text{in}}$  &  $T_{\text{out}}$ ,  $S$  becomes

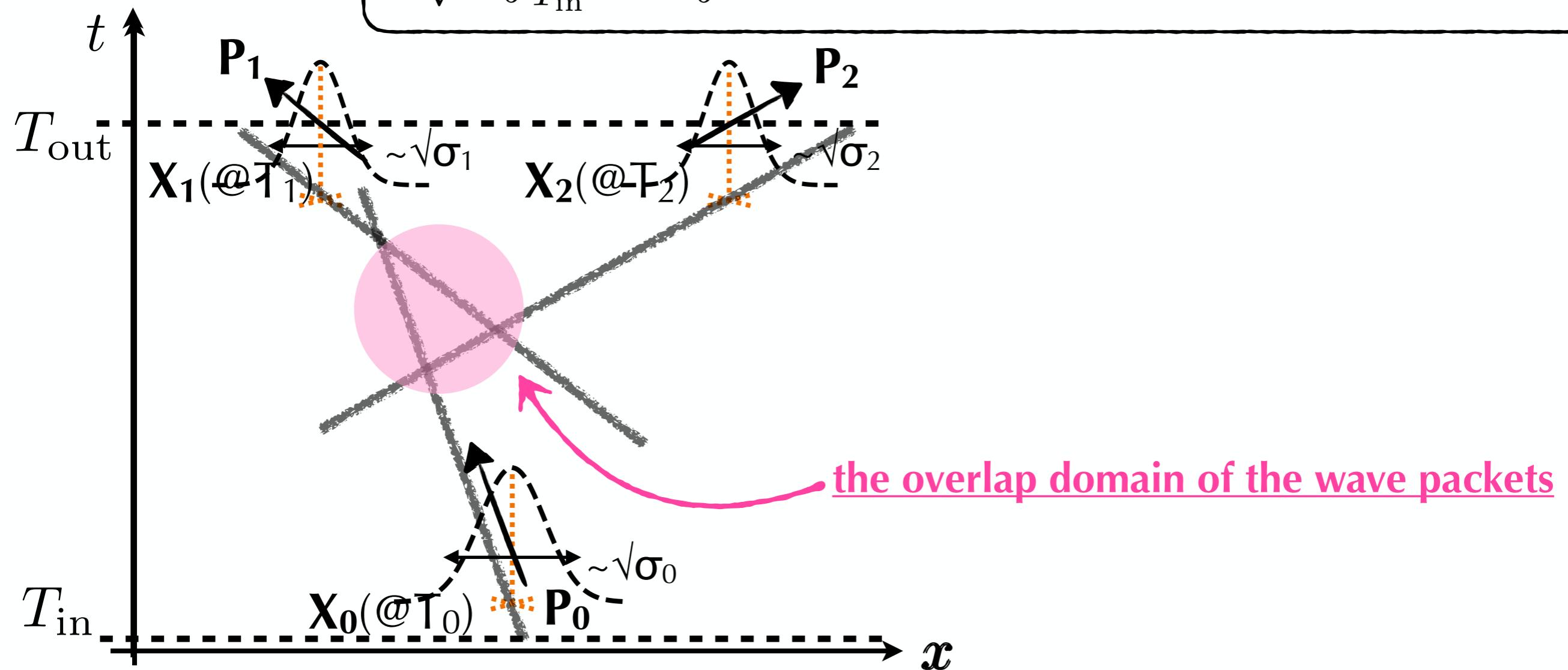
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Wick's theorem  
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# Result of $S(\Phi \rightarrow \phi\phi)$

$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta P)^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$

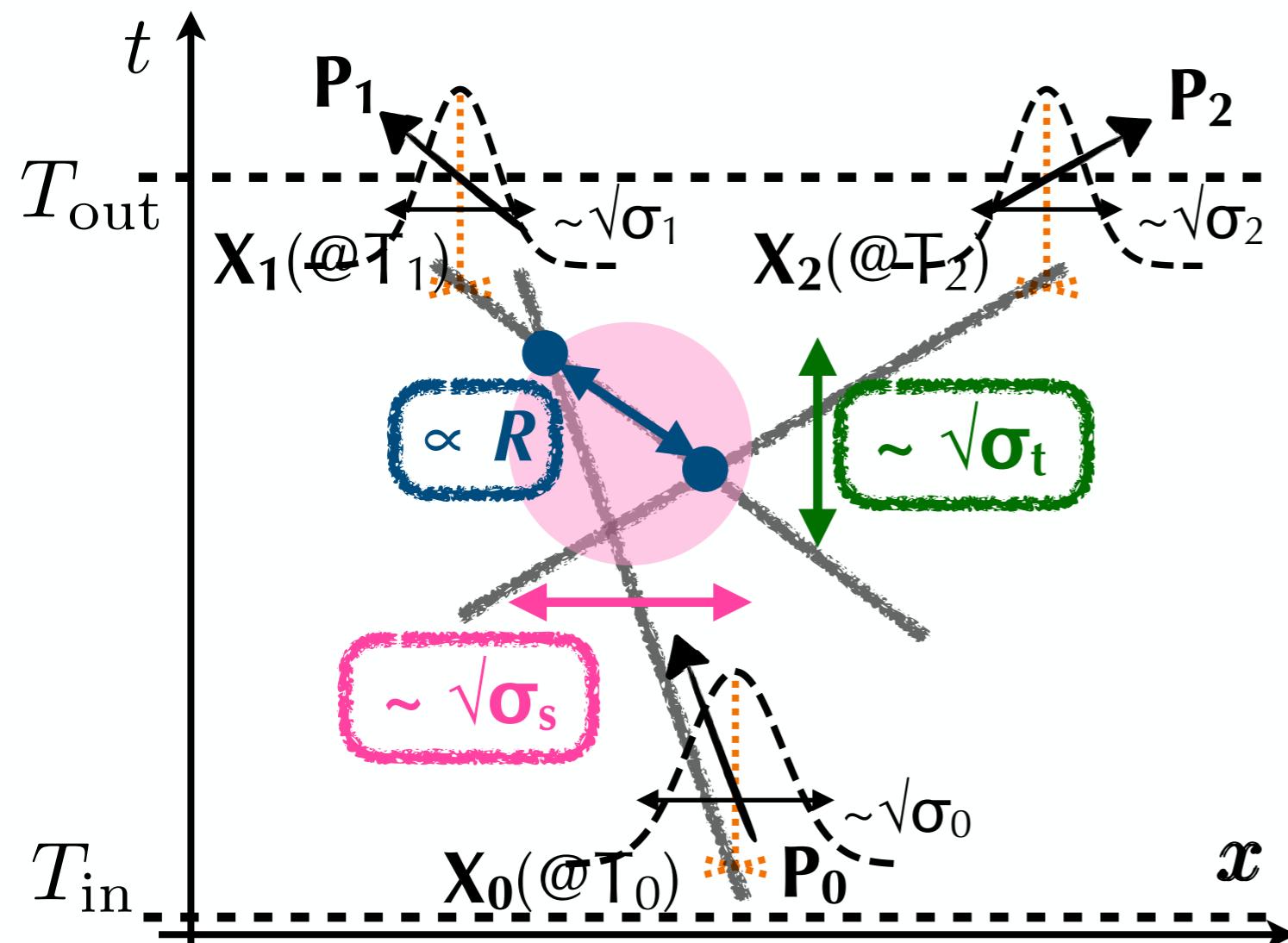
This is the exact analytic form.

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- Feature ①: Geometrical variables characterise  $S$ .

$(\delta\omega \sim \delta E := E_{\text{out}} - E_{\text{in}}, \delta P := P_{\text{out}} - P_{\text{in}})$



# Result of $S(\Phi \rightarrow \phi\phi)$

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- Feature ①: Geometrical variables characterise  $S$ .

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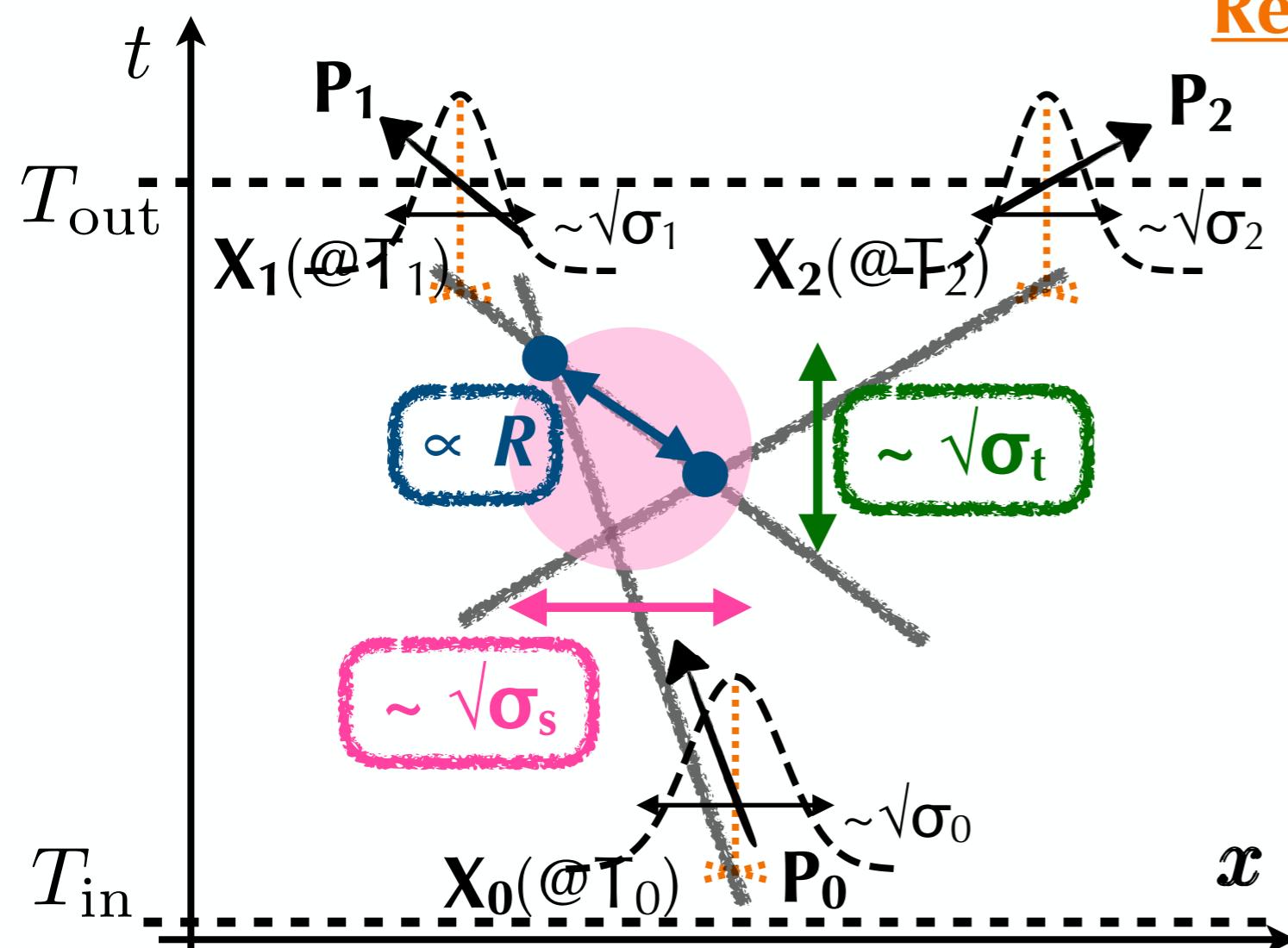
- Feature ②:

The limit ( $\sigma_s \rightarrow \infty$  and  $\sigma_t \rightarrow \infty$ )  $\Rightarrow$

Recovery of the energy-momentum conservation

Note:

$$\left( \sqrt{\frac{\sigma}{2\pi}} e^{-\frac{\sigma}{2}(p-p_0)^2} \xrightarrow[\sigma \rightarrow \infty]{} \delta(p - p_0) \right)$$

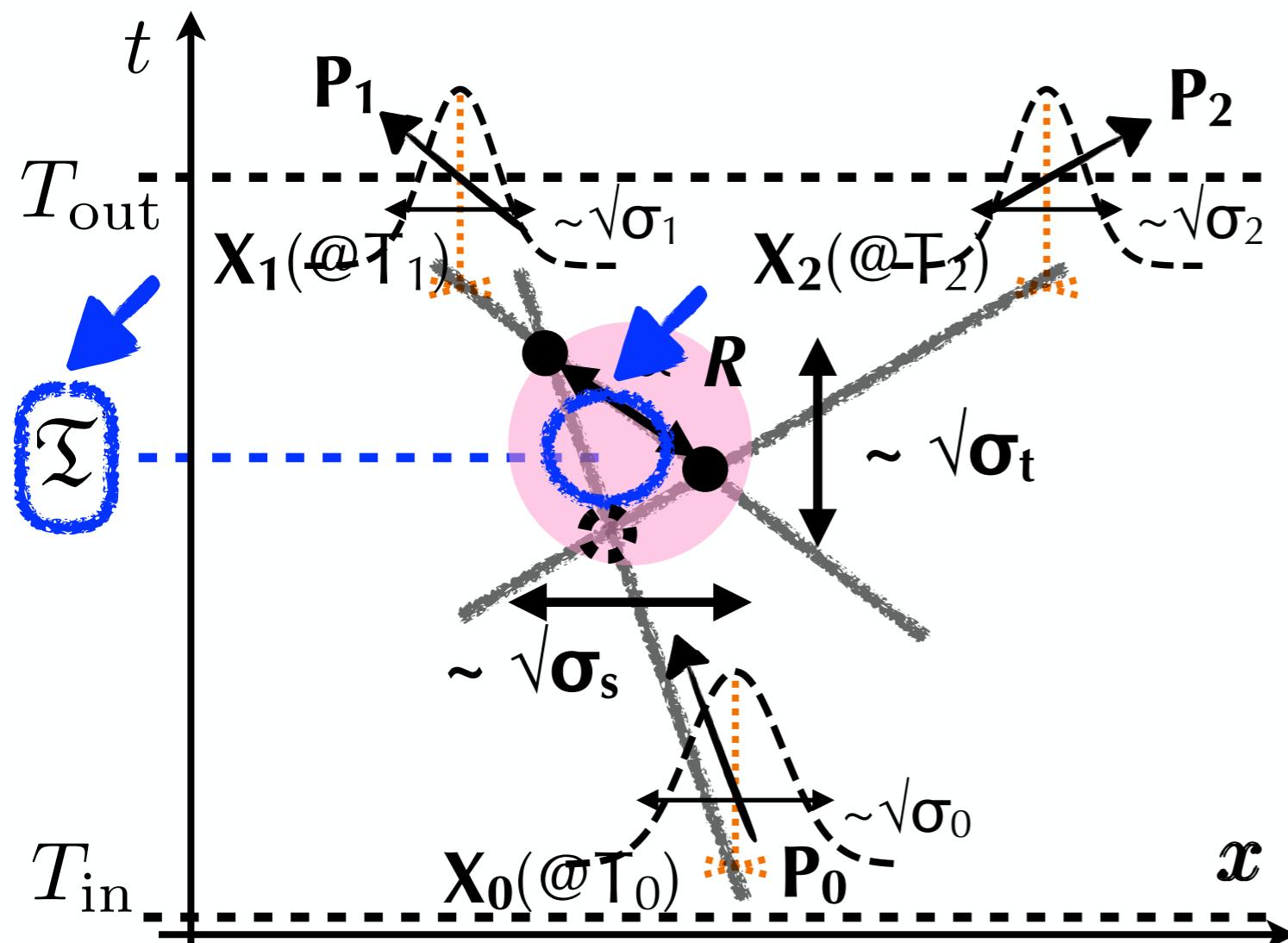


# Result of $S(\Phi \rightarrow \phi\phi)$

$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta P)^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\Sigma)$$

- Feature ③: Terms are classified into “bulk” and “boundary”.

$\Sigma$ : time of overlap (around which three wave packets overlap).



# Result of $S(\Phi \rightarrow \phi\phi)$

$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta P)^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$

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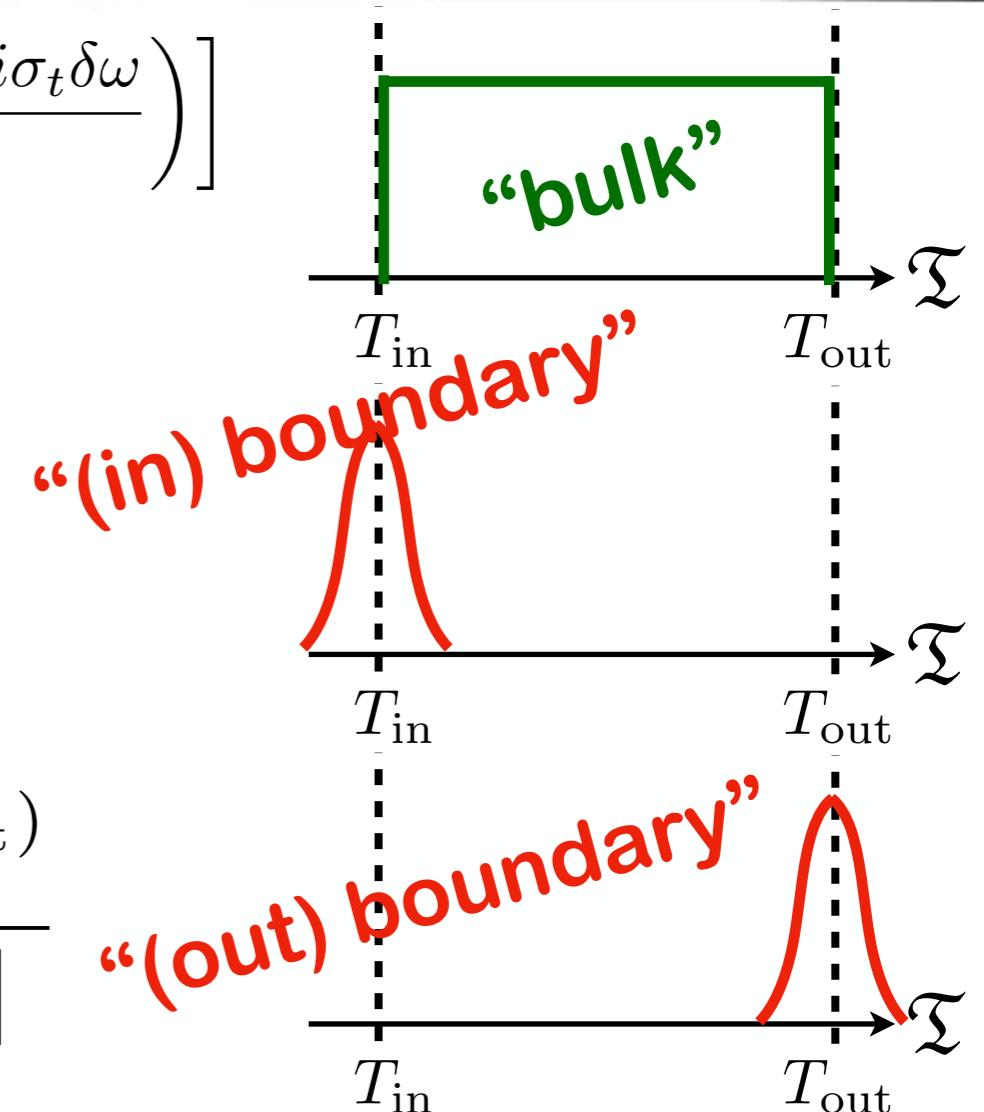
$\mathfrak{T}$ : time of overlap (around which three wave packets overlap).

approximately

$$G(\mathfrak{T}) \sim \frac{1}{2} \left[ \text{sgn} \left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) - \text{sgn} \left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$

$$- \frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t} [\delta\omega - i(\mathfrak{T}-T_{\text{in}})/\sigma_t]}$$

$$+ \frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t} [\delta\omega - i(\mathfrak{T}-T_{\text{out}})/\sigma_t]}$$



# Result of $S(\Phi \rightarrow \phi\phi)$

$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\mathbf{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$

In “1→2”,

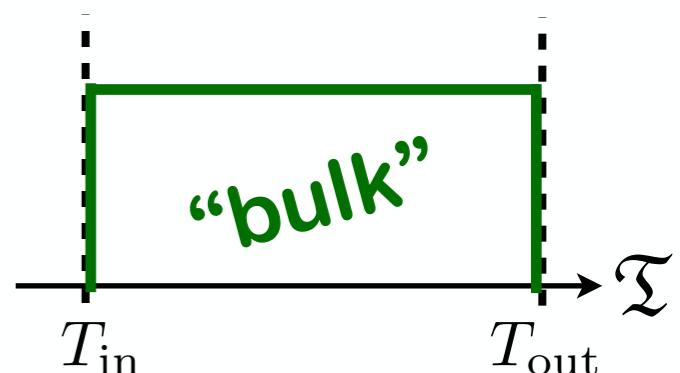
- Bulk part is “time-universal”. As expected, we can show

[Marginalised rate per (Volume) & (Time), from  $S_{\text{bulk}}$  @  $\mathbf{P}_0 \rightarrow \mathbf{0}$ ]  $\xrightarrow{\quad}$  
$$= \left[ \frac{\int d^3X_0 (= \text{in})}{V(T_{\text{out}} - T_{\text{in}})} \int \prod_{j=1,2} \frac{d^3X_j d^3P_j}{(2\pi)^3} |S_{\text{bulk}}|^2 \right]_{P_0 \rightarrow 0}$$

( $\sigma_s \rightarrow \infty$  and  $\sigma_t \rightarrow \infty$ : “plane-wave limit”)

$\Gamma_{\Phi \rightarrow \phi\phi}^{(\text{plane-wave})}$   $\xrightarrow{\quad}$  (the decay width from  $S_{\text{plane-wave}}$ )

$$G(\mathfrak{T}) \supset \frac{1}{2} \left[ \text{sgn} \left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) - \text{sgn} \left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$



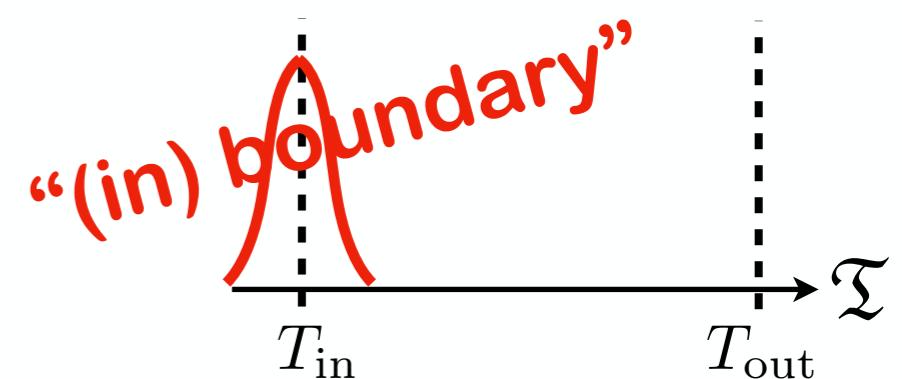
# Result of $S(\Phi \rightarrow \Phi\Phi)$

$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta P)^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$

In "1→2",

- No counterpart of **boundary** terms exists in  $S_{\text{plane-wave}}$ .
- Suppression via energy-non-conservation is **relaxed** as "Exponential" → "Power" [∴ Enhancement].

$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t} [\delta\omega - i(\mathfrak{T}-T_{\text{in}})/\sigma_t]}$$

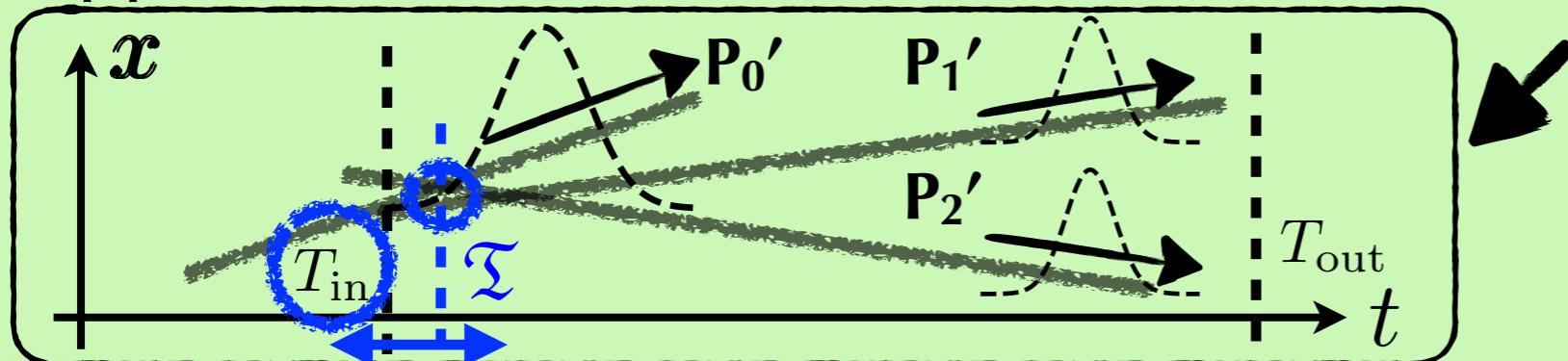


# Result of $S(\Phi \rightarrow \Phi\Phi)$

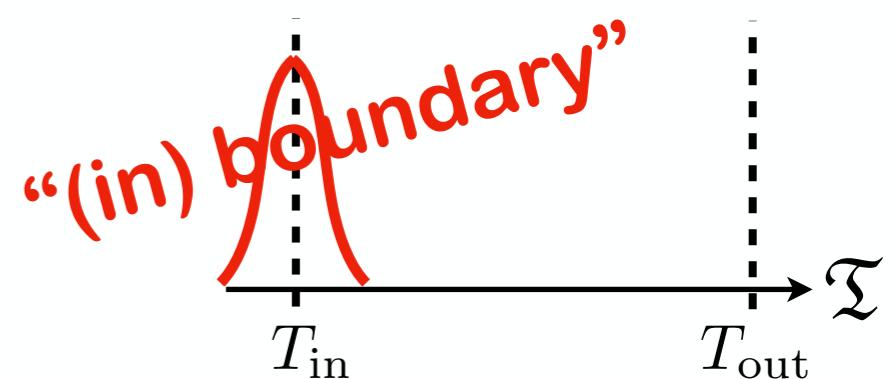
$$S = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta P)^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$

In “1→2”,

- No counterpart of **boundary** terms exists in  $S_{\text{plane-wave}}$ .
- Suppression via energy-non-conservation is **relaxed** as “Exponential” → “Power” [∴ Enhancement].
- Suppression via distances between time domains is **relaxed** e.g., in



$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t^2} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t} [\delta\omega - i(\mathfrak{T}-T_{\text{in}})/\sigma_t]}$$



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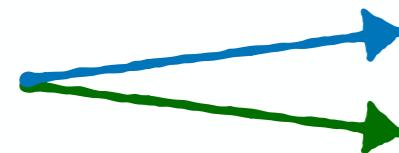
**NEXT**

# Setup of $S(\phi\phi \rightarrow \phi \rightarrow \phi\phi)$

$$S := \langle \overset{\text{free out-state}}{\mathcal{P}_3, \mathcal{P}_4} | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} | \overset{\text{free in-state}}{\mathcal{P}_1, \mathcal{P}_2} \rangle$$

$(\epsilon \simeq M\Gamma)$

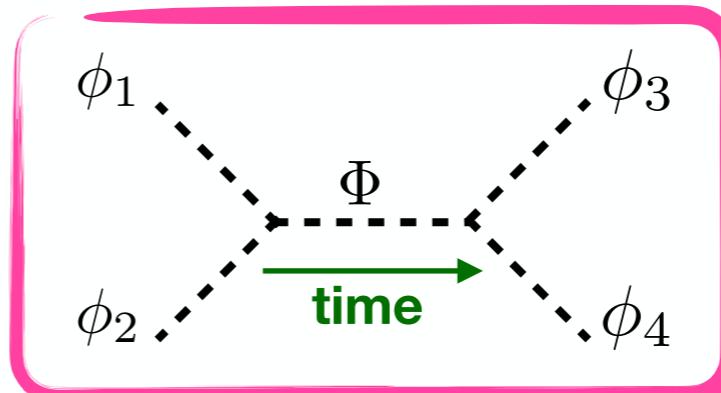
the intermediate part described in simple plane-wave Language



Wick's theorem  
for  $A$  and  $A^+$  (@LO)  
and  
(over)completeness  
of Gaussian basis

$$\begin{aligned} & (-i\kappa)^2 (-i) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + M^2 - i\epsilon} \\ & \times \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3 x f_{\sigma_3; \Pi_3}^*(x) f_{\sigma_4; \Pi_4}^*(x) e^{ip \cdot x} \\ & \times \int_{T_{\text{in}}}^{T_{\text{out}}} dt' \int d^3 x' f_{\sigma_1; \Pi_1}(x') f_{\sigma_2; \Pi_2}(x') e^{-ip \cdot x'} \end{aligned}$$

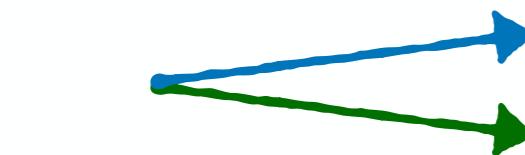
$$\left[ \mathcal{P}_i = \left\{ \sigma_i, \underbrace{X_i^0 (= T_i), \mathbf{X}_i, \mathbf{P}_i}_{=: X_i} \right\} \right] (\Pi_i := \{X_i, \mathbf{P}_i\})$$



# Setup of $S(\phi\phi \rightarrow \phi \rightarrow \phi\phi)$

$$S := \langle \overset{\text{free out-state}}{\mathcal{P}_3, \mathcal{P}_4} | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} \overset{\text{free in-state}}{| \mathcal{P}_1, \mathcal{P}_2 \rangle}$$

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$$(-i\kappa)^2 (-i) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + M^2 - i\epsilon}$$

$$\times \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3 x f_{\sigma_3; \Pi_3}^*(x) f_{\sigma_4; \Pi_4}^*(x) e^{ip \cdot x}$$

$$\times \int_{T_{\text{in}}}^{T_{\text{out}}} dt' \int d^3 x' f_{\sigma_1; \Pi_1}(x') f_{\sigma_2; \Pi_2}(x') e^{-ip \cdot x'}$$

the intermediate part  
described in simple  
plane-wave Language

[Adv.]

a related talk at 4:15PM (@ Seoul)  
TODAY by Mr. WADA Juntaro (Tokyo)

$$\left[ \mathcal{P}_i = \{\sigma_i, \underbrace{X_i^0 (= T_i), X_i, P_i}_{=: X_i}\} \right] (\Pi_i := \{X_i, P_i\})$$

16:15

A complete set of Lorentz-invariant wave packets and modified uncertainty relation

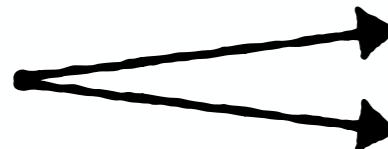
⌚ 15m

We define a set of fully Lorentz-invariant wave packets and show that it spans the corresponding one-particle Hilbert subspace, and hence the whole Fock space as well, with a manifestly Lorentz-invariant completeness relation (resolution of identity). The position-momentum uncertainty relation for this Lorentz-invariant wave packet deviates from the ordinary Heisenberg uncertainty principle, and reduces to it in the non-relativistic limit.

Speaker: WADA Juntaro (The University of Tokyo)

# Setup of $S(\phi\phi \rightarrow \phi \rightarrow \phi\phi)$

$$S := \langle \mathcal{P}_3, \mathcal{P}_4 | T e^{-i \int_{T_{\text{in}}}^{T_{\text{out}}} dt \hat{H}_{\text{int}}^{(I)}(t)} | \mathcal{P}_1, \mathcal{P}_2 \rangle \quad (\epsilon \simeq M\Gamma)$$



Wick's theorem  
for  $A$  and  $A^+$  (@LO)  
and  
(over)completeness  
of Gaussian basis

$$\begin{aligned} & (-i\kappa)^2 (-i) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + M^2 - i\epsilon} \\ & \times \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3 x f_{\sigma_3; \Pi_3}^*(x) f_{\sigma_4; \Pi_4}^*(x) e^{ip \cdot x} \\ & \times \int_{T_{\text{in}}}^{T_{\text{out}}} dt' \int d^3 x' f_{\sigma_1; \Pi_1}(x') f_{\sigma_2; \Pi_2}(x') e^{-ip \cdot x'} \end{aligned}$$

the intermediate part  
described in simple  
plane-wave Language

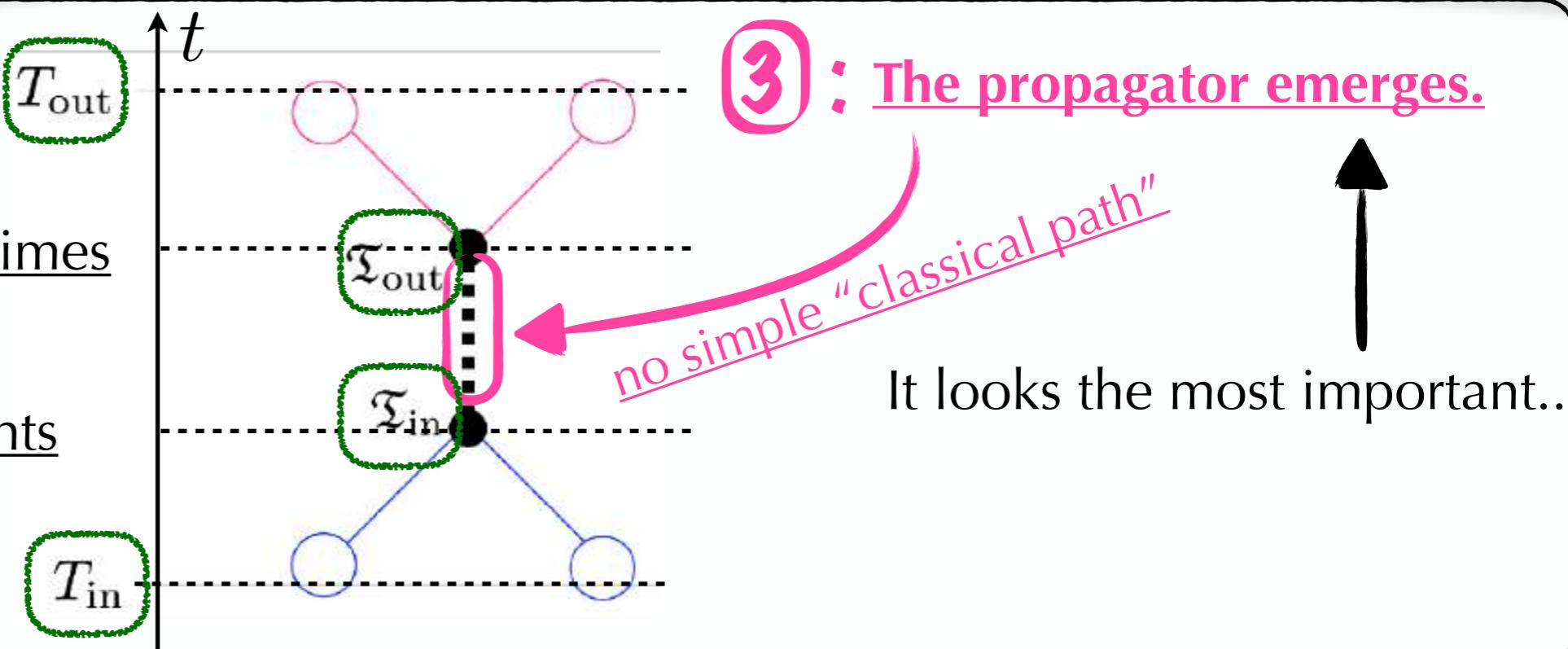
[New feature in  $2 \rightarrow 2$ ]

$T_{\text{out}}$

③

: The propagator emerges.

- ① : four characteristic times in the S-matrix
- ② : two interaction points

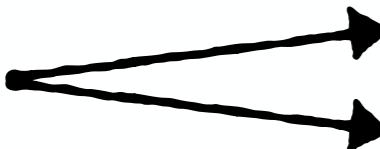


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(free out-state)      (free in-state)

$(\epsilon \simeq M\Gamma)$



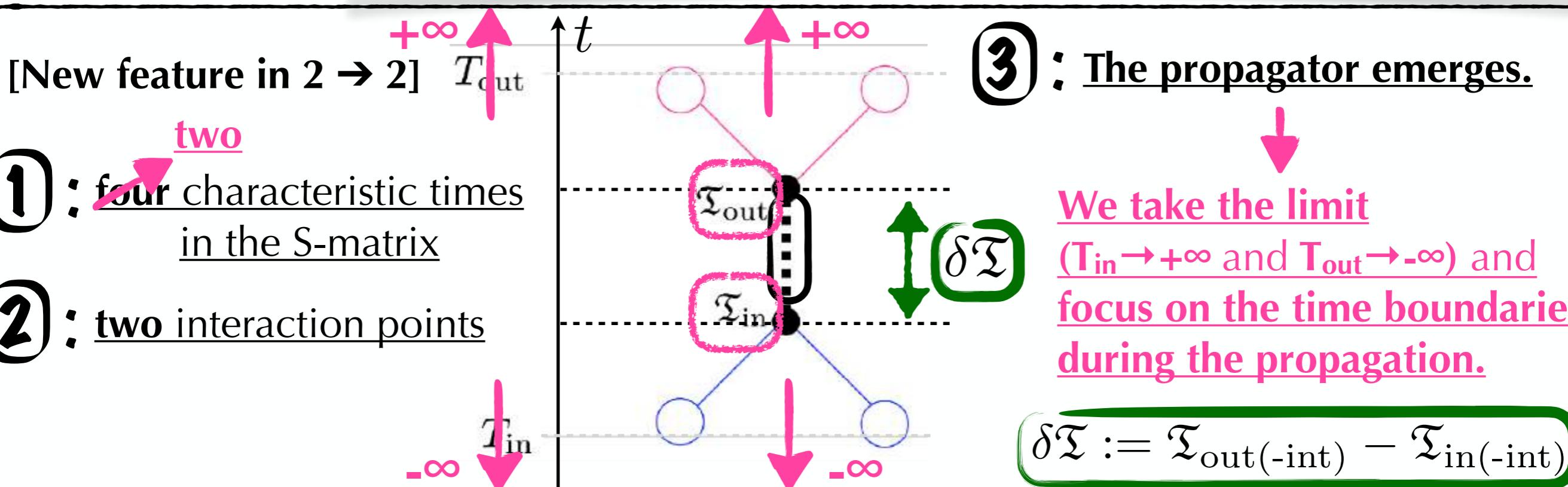
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the intermediate part  
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$$\times \int_{T_{\text{in}}}^{T_{\text{out}}} dt \int d^3 x f_{\sigma_3; \Pi_3}^*(x) f_{\sigma_4; \Pi_4}^*(x) e^{ip \cdot x}$$

$$\times \int_{T_{\text{in}}}^{T_{\text{out}}} dt' \int d^3 x' f_{\sigma_1; \Pi_1}(x') f_{\sigma_2; \Pi_2}(x') e^{-ip \cdot x'}$$



$\downarrow \left( \int_{-\infty (=T_{\text{in}})}^{+\infty (=T_{\text{out}})} dt \int d^3x \right)^2$  “**2→2**” S-matrix structure

focusing on this kernel  $\int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0; \mathbf{p})}$

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^4 \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\text{in}}^3 \sigma_{\text{out}}^3 \zeta_{\text{in}} \zeta_{\text{out}}}$$

$$F(p^0; \mathbf{p}) = F_*(\mathbf{p}) - \frac{\varsigma_+}{2} (p^0 - p_*^0(\mathbf{p}))^2$$

[quadratic for  $p^0$ ,  $(p^0)_*$  is a saddle point]

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focusing on  
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$$\int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0; \mathbf{p})}$$

$$F(p^0; \mathbf{p}) = F_*(\mathbf{p}) - \frac{\varsigma_+}{2} (p^0 - p_*^0(\mathbf{p}))^2$$

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saddle-point  
approximation

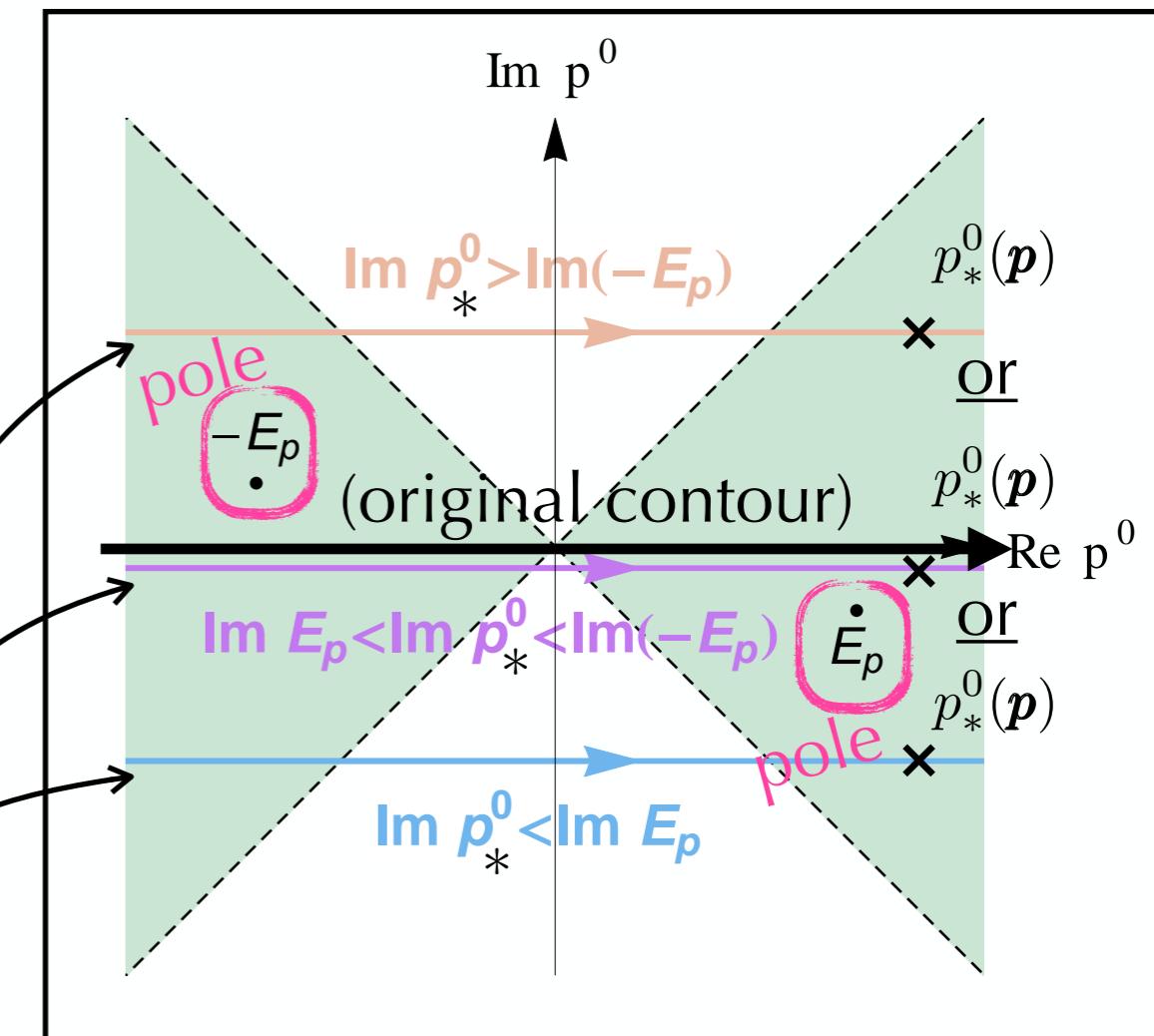
$$\left( \int_{-\infty}^{+\infty} \frac{dp^0}{(2\pi)} \right)$$

Key: Treatment of the poles  
depends on the  $\text{Im}[p_*^0(\mathbf{p})]$

$$\int_{-\infty}^{+\infty} \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{F_*(\mathbf{p})} I_{\text{tot}}(\mathbf{p})$$

$$(E_{\mathbf{p}} = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2})$$

(deformed contour)



$\left( \int_{-\infty}^{+\infty} dt \int d^3x \right)^2$  “**2→2**” S-matrix structure

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^4 \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\text{in}}^3 \sigma_{\text{out}}^3 \sin \varsigma_{\text{out}}}$$

focusing on  
this kernel

$$\int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^F(p^0; p)$$

$$F(p^0; \mathbf{p}) = F_*(\mathbf{p}) - \frac{\varsigma_+}{2} (p^0 - p_*^0(\mathbf{p}))^2$$

[quadratic for  $p^0$ ,  $(p^0)^*$  is a saddle point]

# saddle-point approximation

$$\left( \int_{-\infty}^{+\infty} \frac{dp^0}{(2\pi)} \right)$$

$$\left( E_{\mathbf{p}} = E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2} \right)$$

**Key: Treatment of the poles  
depends on the  $\text{Im}[p_*^0(p)]$**

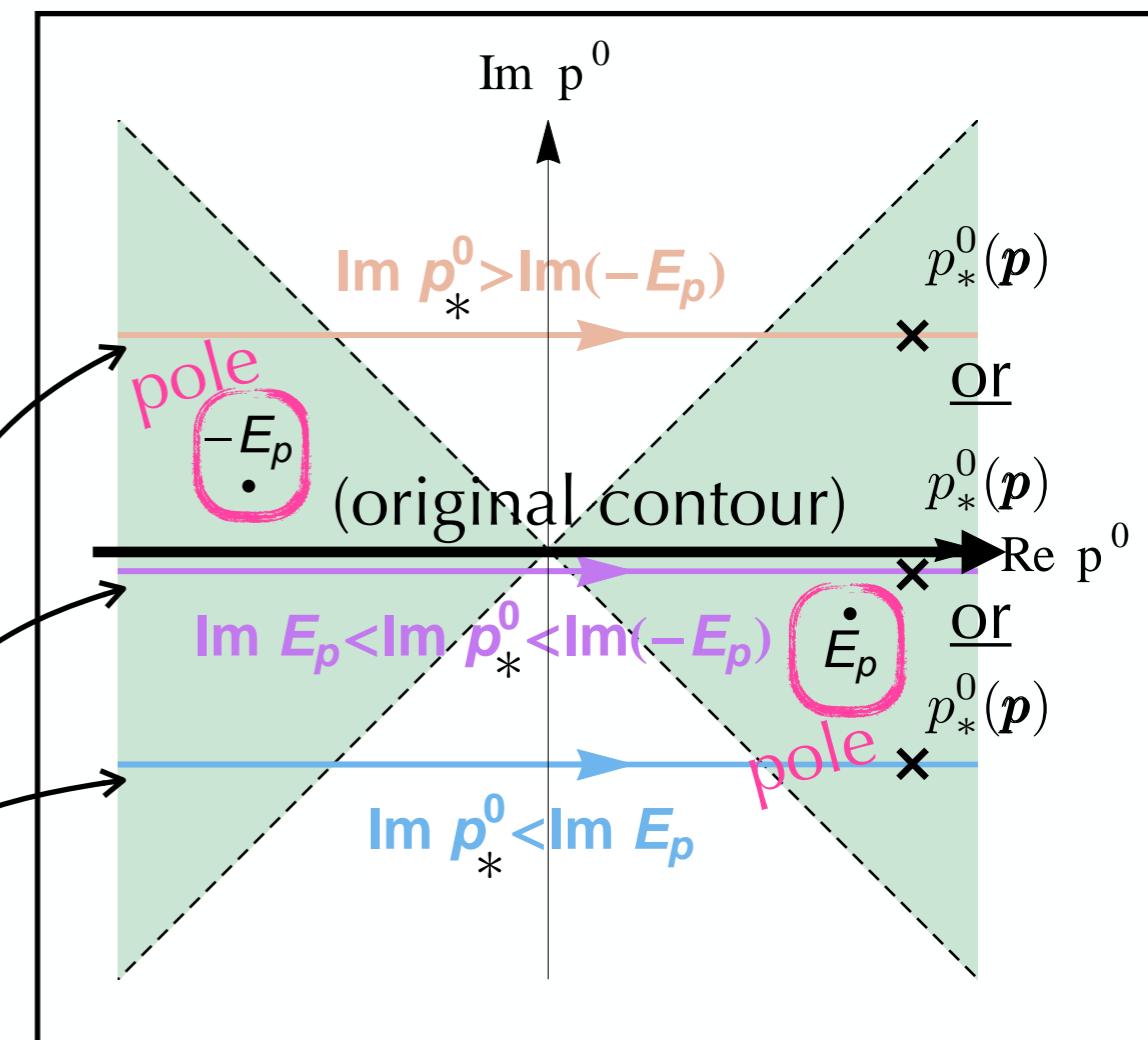
$$\int_{-\infty}^{+\infty} \frac{d^3 p}{(2\pi)^3} e^{F_*(p)} I_{\text{tot}}(p)$$

Im [ $p^*_0(p)$ ]?

$$p_*^0(p) = \omega_\varsigma(p) - i \frac{\delta \mathfrak{T}}{\varsigma +}$$

# dynamical T-product structure (next slide)

positive real



# (Inter.) "Bulk" & "Boundary" in "2→2"



$$I_{\text{tot}}(\mathbf{p}) \simeq \boxed{\begin{aligned} & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) - \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \frac{\delta\mathfrak{T}}{\varsigma_+} + \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} \right) \\ & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) + \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+} \right) \end{aligned}}^{\text{exact}}$$
  

$$\boxed{+ \frac{1}{\sqrt{2\pi\varsigma_+}} \frac{-i}{-\left( \omega_\varsigma(\mathbf{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2 + E^2(\mathbf{p}) - i\epsilon}}}_{\substack{\text{saddle-point} \\ \text{approximated}}}$$

# (Inter.) "Bulk" & "Boundary" in "2→2"



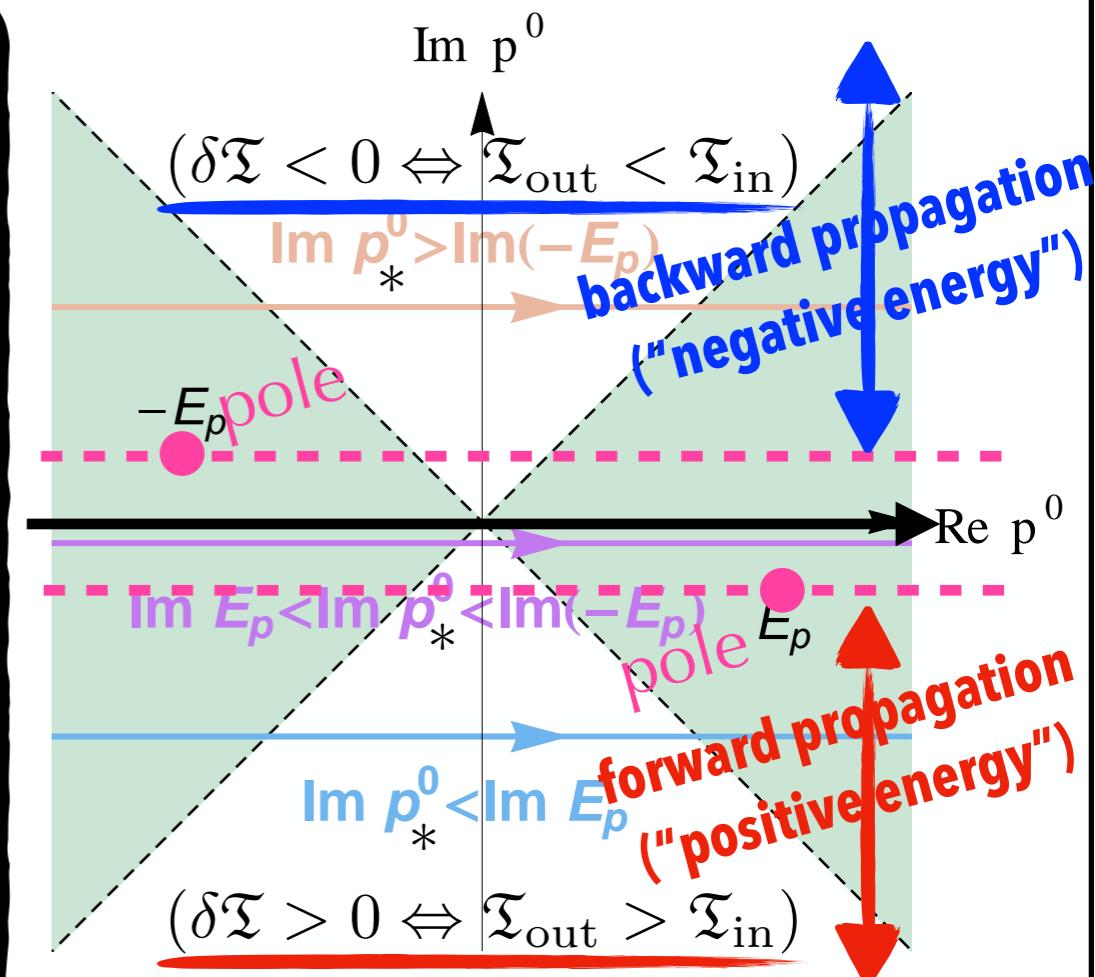
$$I_{\text{tot}}(\mathbf{p}) \simeq \begin{aligned} & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) - \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\Im}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \frac{\delta\Im}{\varsigma_+} + \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} \right) \\ (\omega_\varsigma(\mathbf{p}) > 0) & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) + \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\Im}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} - \frac{\delta\Im}{\varsigma_+} \right) \\ & + \frac{1}{\sqrt{2\pi\varsigma_+}} \frac{-i}{-\left( \omega_\varsigma(\mathbf{p}) - i\frac{\delta\Im}{\varsigma_+} \right)^2 + E^2(\mathbf{p}) - i\epsilon} \end{aligned}$$

considered as  
“inter. bulk”

This is because...

- The **causal structure** is the **same** with that of the (plane-wave) **Feynman Propagator**.
- We can check that the **energy-momentum conservation** is **recovered** in the limit ( $\epsilon \rightarrow 0$ ,  $\sigma_{\text{in}} \rightarrow \infty$  &  $\sigma_{\text{out}} \rightarrow \infty$ ). ( $\epsilon \simeq M\Gamma$ )  
 $[\sigma_{\text{in}}/\sigma_{\text{out}} = \text{spacial-averaged sizes of in-/out-wave packets}]$
- Note: naively Lorentzian  $\Rightarrow$  Gaussian resonance, further deformations comes in  $\int d^3p$ .

PRELIMINARY



# (Inter.) "Bulk" & "Boundary" in "2→2"



$$I_{\text{tot}}(\mathbf{p}) \simeq + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) - \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \frac{\delta\mathfrak{T}}{\varsigma_+} + \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} \right)$$

$$+ \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) + \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+} \right)$$

$$+ \frac{1}{\sqrt{2\pi\varsigma_+}} \frac{-i}{-\left( \omega_\varsigma(\mathbf{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2 + E^2(\mathbf{p}) - i\epsilon}$$

considered as  
"inter. boundary"

This is because...

- Compared with the "bulk",

(1) absent of the **energy-suppression factor**:

*relatively* →

as we observed in "1→2"

$$\times e^{+\frac{\varsigma_+}{2} (\omega_\varsigma(\mathbf{p}) - E(\mathbf{p}))^2}$$

(2) absent of the **" $\delta\mathfrak{T}$ "-enhancement factor**:

→

$$\times e^{-\frac{1}{2\varsigma_+} (\delta\mathfrak{T})^2}$$

- Full energy conservation is NOT recovered even in the limit

$(\epsilon \rightarrow 0, \sigma_{\text{in}} \rightarrow \infty \text{ & } \sigma_{\text{out}} \rightarrow \infty)$ , due to  $\omega_\varsigma(\mathbf{p}_*) \neq E(\mathbf{p}_*)$  [under the limit]

$$(\omega_\varsigma(\mathbf{p}_*) \rightarrow E_{\text{in}} = E_{\text{out}}, E(\mathbf{p}_*) \rightarrow E(P_{\text{in}}) = E(P_{\text{out}}))$$

**PRELIMINARY**

# (Inter.) "Bulk" & "Boundary" in "2→2"

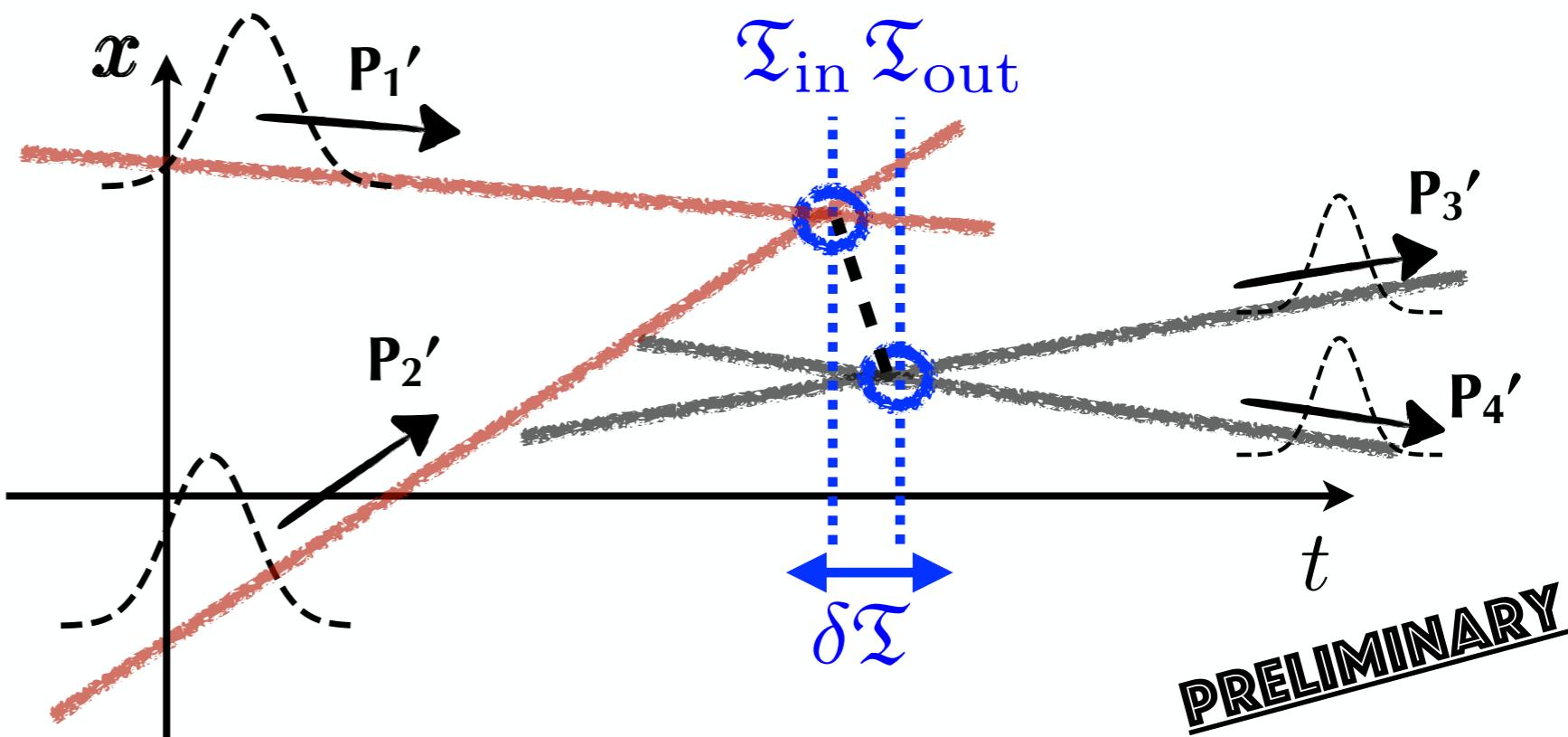


$$\begin{aligned}
 I_{\text{tot}}(\mathbf{p}) \simeq & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) - \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \frac{\delta\mathfrak{T}}{\varsigma_+} + \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} \right) \\
 & + \frac{e^{-\frac{\varsigma_+}{2}} \left( \omega_\varsigma(\mathbf{p}) + \sqrt{E^2(\mathbf{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2}{2\sqrt{E^2(\mathbf{p}) - i\epsilon}} \theta \left( \Im \sqrt{E^2(\mathbf{p}) - i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+} \right) \\
 & + \boxed{\frac{1}{\sqrt{2\pi\varsigma_+}} \frac{-i}{-\left( \omega_\varsigma(\mathbf{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+} \right)^2 + E^2(\mathbf{p}) - i\epsilon}}
 \end{aligned}$$

considered as  
"inter. boundary"

This is because...

- $\delta\mathfrak{T}$  can be small e.g., in the configuration of the in- & out-states:



# Summary & Discussion

1. The S-matrix in Gaussian wave packet contains **full information** of the **quantum particles**. → **More informative & regularised**.
2. (Classical) trajectories of in-/out-states play significant roles.  
→ Characterising S-matrix, in particular “**bulk**” and “**boundary**”.
3. The “bulk”-“boundary” structure is also found in the intermediate (off-shell) state of ‘ $2 \rightarrow 2$ ’. → Appropriate time-ordering in bulk, also.

[discussion/what I would like to do in future]

- more details on geometric structure of ‘ $2 \rightarrow 2$ ’
- concrete general form of the ‘ $2 \rightarrow 2$ ’ S-matrix
- implications on ‘ $1 \rightarrow 2$ ’ as the latter part of ‘ $2 \rightarrow 2$ ’
- generic features of the probabilities of ‘ $2 \rightarrow 2$ ’
- various theoretical points of the wave-packet formalism
- applications for various phenomena

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THANK YOU!

# **BACKUP SLIDES**

# Intro: S-matrix in plane-wave basis

[QFT textbooks]

- The **standard tool** for describing quantum processes of **particles**:

• Basis (@ Schrödinger Pic.):  $e^{i \mathbf{p} \cdot \mathbf{x}}$

(plane wave: the eigenstate of  $\mathbf{p}$ )  $\leftrightarrow$   $\mathbf{x}$  completely undetermined

• Form:  $S = (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) (i\mathcal{M})$

Energy-momentum conservation    
is manifest (due to translational Inv.)   easy to be constructed  
by Feynman rules

• established formalism, very very widely used in Physics

- On the other hand...,

[(Volume)(Time)  $\rightarrow \infty$ ]

•  $|S|^2$  is ill-defined due to  $|\delta^4(P_{\text{out}} - P_{\text{in}})|^2 = \delta^4(P_{\text{out}} - P_{\text{in}}) \times \underline{\delta^4(0)}$ .

$\Rightarrow$  Only the averaged (per V and T) frequencies of events is calculable.

 decay widths & cross sections

•  $T_{\text{in}} (= T_{\text{initial}}) = -\infty$ , and  $T_{\text{out}} (= T_{\text{final}}) = +\infty$

• This is because the plane wave is a **non-normalisable** mode.

# Details on $S(\Phi \rightarrow \phi\phi)$

[Ishikawa, Oda (1809.04285)]

- $$S = \frac{i\kappa}{\sqrt{2}} \left( \prod_A (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\mathbf{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t} G(\mathfrak{T})$$
  - $$\circ \quad G(\mathfrak{T}) := \int_{T_{\text{in}}}^{T_{\text{out}}} \frac{dt}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t-\mathfrak{T}-i\sigma_t\delta\omega)^2}$$

$$= \frac{1}{2} \left[ \operatorname{erf}\left(\frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \operatorname{erf}\left(\frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) \right]$$

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$$

- $$E_A := \sqrt{m_A^2 + \mathbf{P}_A^2}$$
- $$\mathbf{V}_A := \frac{\mathbf{P}_A}{E_A}$$
- $$\sigma_s := \left( \sum_{A=0}^2 \frac{1}{\sigma_A} \right)^{-1}$$
- $$\sigma_t := \frac{\sigma_s}{\Delta V^2}$$
- $$\mathfrak{T} := \sigma_t \frac{\overline{\mathbf{V}} \cdot \overline{\mathbf{\mathfrak{X}}} - \overline{\mathbf{V}} \cdot \overline{\mathbf{\mathfrak{X}}}}{\sigma_s} = \frac{\overline{\mathbf{V}} \cdot \overline{\mathbf{\mathfrak{X}}} - \overline{\mathbf{V}} \cdot \overline{\mathbf{\mathfrak{X}}}}{\Delta V^2}$$
- $$\mathcal{R} := \frac{\Delta \mathbf{\mathfrak{X}}^2}{\sigma_s} - \frac{\mathfrak{T}^2}{\sigma_t}$$

$$\overline{\mathbf{Q}} := \sigma_s \sum_A \frac{\mathbf{Q}_A}{\sigma_A}. \quad \Delta \mathbf{Q}^2 := \overline{\mathbf{Q}^2} - \overline{\mathbf{Q}}^2$$

$$\mathbf{\mathfrak{X}}_A := \mathbf{X}_A - \mathbf{V}_A T_A \quad [\mathbf{\mathfrak{X}}_A = \mathbf{\Xi}_A(0)]$$

$$\begin{aligned} \delta \mathbf{P} &:= \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_0 \\ \delta E &:= E_1 + E_2 - E_0 \\ \delta \omega &:= \delta E - \overline{\mathbf{V}} \cdot \delta \mathbf{P} \end{aligned}$$

# Details on $S(\Phi \rightarrow \phi\phi)$

[Ishikawa, Oda (1809.04285)]

- $\sigma_t = \frac{1}{\sigma_s} \left[ \frac{(\delta V_1)^2}{\sigma_0 \sigma_1} + \frac{(\delta V_2)^2}{\sigma_0 \sigma_2} + \frac{(\delta V_1 - \delta V_2)^2}{\sigma_1 \sigma_2} \right]^{-1}, \quad \boxed{\delta Q_a := Q_a - Q_0}$
- $\mathfrak{T} = -\sigma_s \sigma_t \left[ \frac{\delta \mathfrak{X}_1 \cdot \delta V_1}{\sigma_0 \sigma_1} + \frac{\delta \mathfrak{X}_2 \cdot \delta V_2}{\sigma_0 \sigma_2} + \frac{(\delta \mathfrak{X}_1 - \delta \mathfrak{X}_2) \cdot (\delta V_1 - \delta V_2)}{\sigma_1 \sigma_2} \right],$
- $\mathcal{R} = \sigma_s \left\{ \frac{(\delta \mathfrak{X}_1)^2}{\sigma_0 \sigma_1} + \frac{(\delta \mathfrak{X}_2)^2}{\sigma_0 \sigma_2} + \frac{(\delta \mathfrak{X}_1 - \delta \mathfrak{X}_2)^2}{\sigma_1 \sigma_2} - \sigma_s \sigma_t \left[ \frac{\delta \mathfrak{X}_1 \cdot \delta V_1}{\sigma_0 \sigma_1} + \frac{\delta \mathfrak{X}_2 \cdot \delta V_2}{\sigma_0 \sigma_2} + \frac{(\delta \mathfrak{X}_1 - \delta \mathfrak{X}_2) \cdot (\delta V_1 - \delta V_2)}{\sigma_1 \sigma_2} \right]^2 \right\}.$

# Details on $S(\Phi\Phi \rightarrow \Phi \rightarrow \Phi\Phi)$

[Ishiwaka, KN, Oda  
(2006.14159, 2102.12032)]

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^4 \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\text{in}}^3 \sigma_{\text{out}}^3 \varsigma_{\text{in}} \varsigma_{\text{out}}} \\ \times \boxed{\int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0; \mathbf{p})}}$$

$$F(p^0; \mathbf{p}) = F_*(\mathbf{p}) - \frac{\varsigma_+}{2} (p^0 - p_*^0(\mathbf{p}))^2$$

$$F_*(\mathbf{p}) = \left( -\frac{\mathcal{R}_{\text{in}}}{2} - \frac{\sigma_{\text{in}}}{2} (\mathbf{p} - \mathbf{P}_{\text{in}})^2 - i\bar{\Xi}_{\text{in}} \cdot (\mathbf{p} - \mathbf{P}_{\text{in}}) \right) + \left( -\frac{\mathcal{R}_{\text{out}}}{2} - \frac{\sigma_{\text{out}}}{2} (\mathbf{p} - \mathbf{P}_{\text{out}})^2 + i\bar{\Xi}_{\text{out}} \cdot (\mathbf{p} - \mathbf{P}_{\text{out}}) \right) \\ - \frac{(\delta\mathfrak{T})^2}{2\varsigma_+} + i\varsigma \left( \frac{\mathfrak{T}_{\text{in}}}{\varsigma_{\text{in}}} + \frac{\mathfrak{T}_{\text{out}}}{\varsigma_{\text{out}}} \right) \delta\omega(\mathbf{p}) - \frac{\varsigma}{2} (\delta\omega(\mathbf{p}))^2, \quad \boxed{\Xi_a := \mathbf{X}_a - \mathbf{V}_a X_a^0 \quad [\Xi_a = \Xi_A(0)]}$$

- $p_*^0(\mathbf{p}) = \omega_\varsigma(\mathbf{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+} \leftarrow \boxed{\omega_\varsigma(\mathbf{p}) := \frac{\varsigma_{\text{in}}\omega_{\text{in}}(\mathbf{p}) + \varsigma_{\text{out}}\omega_{\text{out}}(\mathbf{p})}{\varsigma_{\text{in}} + \varsigma_{\text{out}}} \quad \delta\mathfrak{T} := \mathfrak{T}_{\text{out-int}} - \mathfrak{T}_{\text{in-int}}}$

- $\sigma_{\text{in}} := \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2}$  and  $\sigma_{\text{out}} := \frac{\sigma_3\sigma_4}{\sigma_3 + \sigma_4}$  ( $\varsigma_{\text{in}} = \frac{\sigma_1 + \sigma_2}{(\mathbf{V}_1 - \mathbf{V}_2)^2}$  and  $\varsigma_{\text{out}} = \frac{\sigma_3 + \sigma_4}{(\mathbf{V}_3 - \mathbf{V}_4)^2}$ )  
[similar definitions for "out" variables]

- $\varsigma_+ := \varsigma_{\text{in}} + \varsigma_{\text{out}}, \quad \varsigma := \left( \frac{1}{\varsigma_{\text{in}}} + \frac{1}{\varsigma_{\text{out}}} \right)^{-1} \quad \boxed{\bar{\mathbf{Q}}_{\text{in}} := \sigma_{\text{in}} \left( \frac{\mathbf{Q}_1}{\sigma_1} + \frac{\mathbf{Q}_2}{\sigma_2} \right), \quad \Delta\mathbf{Q}_{\text{in}}^2 := \overline{\mathbf{Q}^2}_{\text{in}} - \bar{\mathbf{Q}}_{\text{in}}^2}$

$$\mathfrak{T}_{\text{in}} := \frac{\overline{\mathbf{V}}_{\text{in}} \cdot \overline{\mathfrak{X}}_{\text{in}} - \overline{\mathfrak{X}} \cdot \overline{\mathbf{V}}_{\text{in}}}{\Delta V_{\text{in}}^2}, \quad \mathcal{R}_{\text{in}} := \frac{\Delta\mathfrak{X}_{\text{in}}^2}{\sigma_{\text{in}}} - \frac{\mathfrak{T}_{\text{in}}^2}{\varsigma_{\text{in}}}$$

# Details on $S(\Phi\Phi \rightarrow \Phi \rightarrow \Phi\Phi)$ [Ishiwaka, KN, Oda (2006.14159, 2102.12032)]

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^4 \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\text{in}}^3 \sigma_{\text{out}}^3 \varsigma_{\text{in}} \varsigma_{\text{out}}} \\ \times \boxed{\int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0; \mathbf{p})}}$$

$$F(p^0; \mathbf{p}) = F_*(\mathbf{p}) - \frac{\varsigma_+}{2} (p^0 - p_*^0(\mathbf{p}))^2$$

$$F_*(\mathbf{p}) = \left( -\frac{\mathcal{R}_{\text{in}}}{2} - \frac{\sigma_{\text{in}}}{2} (\mathbf{p} - \mathbf{P}_{\text{in}})^2 - i\bar{\Xi}_{\text{in}} \cdot (\mathbf{p} - \mathbf{P}_{\text{in}}) \right) + \left( -\frac{\mathcal{R}_{\text{out}}}{2} - \frac{\sigma_{\text{out}}}{2} (\mathbf{p} - \mathbf{P}_{\text{out}})^2 + i\bar{\Xi}_{\text{out}} \cdot (\mathbf{p} - \mathbf{P}_{\text{out}}) \right) \\ - \frac{(\delta\mathfrak{T})^2}{2\varsigma_+} + i\varsigma \left( \frac{\mathfrak{T}_{\text{in}}}{\varsigma_{\text{in}}} + \frac{\mathfrak{T}_{\text{out}}}{\varsigma_{\text{out}}} \right) \delta\omega(\mathbf{p}) - \frac{\varsigma}{2} (\delta\omega(\mathbf{p}))^2, \quad \boxed{\Xi_a := \mathbf{X}_a - \mathbf{V}_a X_a^0 \quad [\Xi_a = \Xi_A(0)]}$$

$$\omega_{\text{in}}(\mathbf{p}) := E_{\text{in}} + \overline{\mathbf{V}}_{\text{in}} \cdot (\mathbf{p} - \mathbf{P}_{\text{in}}) \\ \omega_{\text{out}}(\mathbf{p}) := E_{\text{out}} + \overline{\mathbf{V}}_{\text{out}} \cdot (\mathbf{p} - \mathbf{P}_{\text{out}})$$

$$\overline{\mathbf{V}}_{\text{in}} := \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \left( \frac{\mathbf{V}_1}{\sigma_1} + \frac{\mathbf{V}_2}{\sigma_2} \right) \\ \overline{\mathbf{V}}_{\text{out}} := \frac{\sigma_3\sigma_4}{\sigma_3 + \sigma_4} \left( \frac{\mathbf{V}_3}{\sigma_3} + \frac{\mathbf{V}_4}{\sigma_4} \right)$$

$$\mathfrak{T}_{\text{in-int}} = -\frac{(\mathbf{V}_1 - \mathbf{V}_2) \cdot (\Xi_1 - \Xi_2)}{(\mathbf{V}_1 - \mathbf{V}_2)^2}, \\ \mathfrak{T}_{\text{out-int}} = -\frac{(\mathbf{V}_3 - \mathbf{V}_4) \cdot (\Xi_3 - \Xi_4)}{(\mathbf{V}_3 - \mathbf{V}_4)^2}.$$

$$\bar{\Xi}_{\text{in}} = \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \left( \frac{\Xi_1}{\sigma_1} + \frac{\Xi_2}{\sigma_2} \right) \rightsquigarrow \frac{\Xi_1 + \Xi_2}{2}, \\ \bar{\Xi}_{\text{out}} = \frac{\sigma_3\sigma_4}{\sigma_3 + \sigma_4} \left( \frac{\Xi_3}{\sigma_3} + \frac{\Xi_4}{\sigma_4} \right) \rightsquigarrow \frac{\Xi_3 + \Xi_4}{2};$$

$$\mathcal{R}_{\text{in}} = \frac{(\Xi_1 - \Xi_2)^2 + [\hat{\mathbf{V}}_{12} \cdot (\Xi_1 - \Xi_2)]^2}{\sigma_1 + \sigma_2}, \\ \mathcal{R}_{\text{out}} = \frac{(\Xi_3 - \Xi_4)^2 + [\hat{\mathbf{V}}_{34} \cdot (\Xi_3 - \Xi_4)]^2}{\sigma_3 + \sigma_4}, \\ \text{in which } \hat{\mathbf{V}}_{12} := \frac{\mathbf{V}_1 - \mathbf{V}_2}{|\mathbf{V}_1 - \mathbf{V}_2|} \text{ and } \hat{\mathbf{V}}_{34} := \frac{\mathbf{V}_3 - \mathbf{V}_4}{|\mathbf{V}_3 - \mathbf{V}_4|}.$$

# Cross check of $S(2 \rightarrow 2)$ : thimble decomposition

- $$I(\mathbf{p}) := \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{-i}{-(p^0)^2 + E_{\mathbf{p}}^2} e^{-\frac{\varsigma_+}{2}(p^0 - p_*^0(\mathbf{p}))^2} = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} e^{\mathcal{F}(p^0; \mathbf{p})}$$

$$E_{\mathbf{p}} := (E_{\mathbf{p}}^2 - i\epsilon)^{1/2}$$

$$\mathcal{F}(p^0) = -\frac{\varsigma_+}{2} (p^0 - p_*^0)^2 - \ln(-(p^0)^2 + E_{\mathbf{p}}^2)$$

$$\Im(\mathcal{F}(p^0) - \mathcal{F}_{(i)}) = 0$$

(three saddle points)

$$p_{(*)}^0 = p_*^0 + \frac{1}{\varsigma_+ - (p_*^0)^2 + E_{\mathbf{p}}^2} + \dots,$$

$$p_{(\pm)}^0 = \pm E_{\mathbf{p}} + \frac{1}{\varsigma_+} \frac{1}{p_*^0 \mp E_{\mathbf{p}}} + \dots,$$

**Anti-thimble** (steepest ascent path) for  $\mathbf{p}^0_{(i)}$

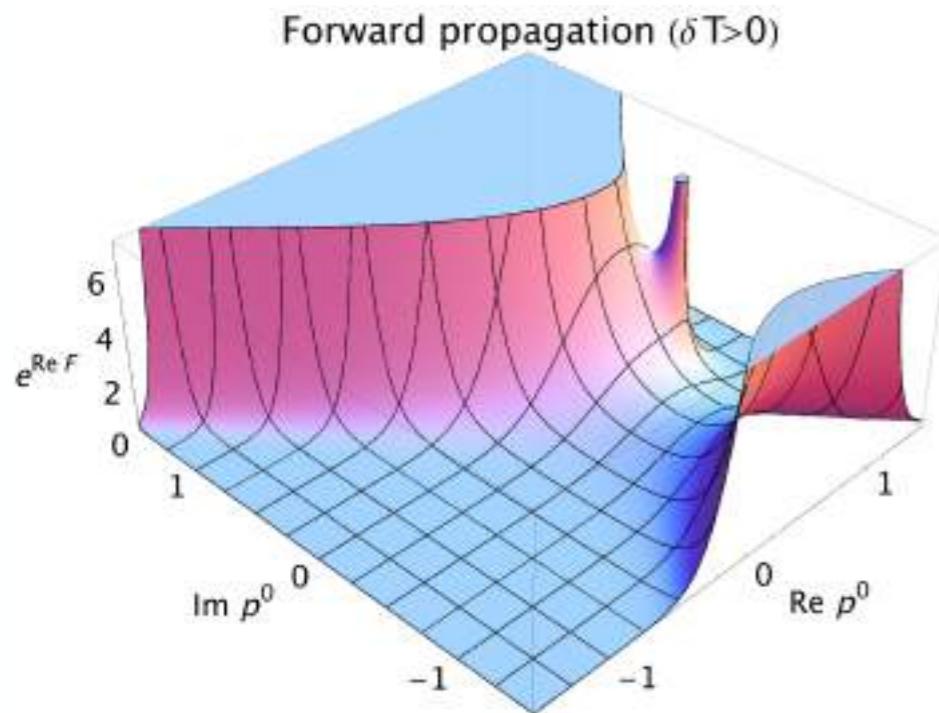
- $$I = \sum_{i=*, \pm} \underbrace{\langle \mathcal{K}_{(i)}, \mathbb{R} \rangle}_{\substack{\text{intersection number} \\ \text{under} \\ \varsigma_+ \gg 1}} I_{(i)} \longrightarrow I_{(i)} = \int_{\mathcal{J}_{(i)}} \frac{dp^0}{2\pi i} e^{\mathcal{F}(p^0)}$$

**Lefschetz thimble** (steepest decent path) for  $\mathbf{p}^0_{(i)}$

- $$I_{(*)} \underset{\varsigma_+ \gg 1}{\sim} \frac{1}{\sqrt{2\pi\varsigma_+}} \frac{-i}{-(p_*^0)^2 + E_{\mathbf{p}}^2}, \quad I_{(\pm)} \underset{\varsigma_+ \gg 1}{\sim} \frac{e}{\sqrt{2\pi}} \frac{e^{-\frac{\varsigma_+}{2}(p_*^0 \mp E_{\mathbf{p}})^2}}{2E_{\mathbf{p}}}.$$

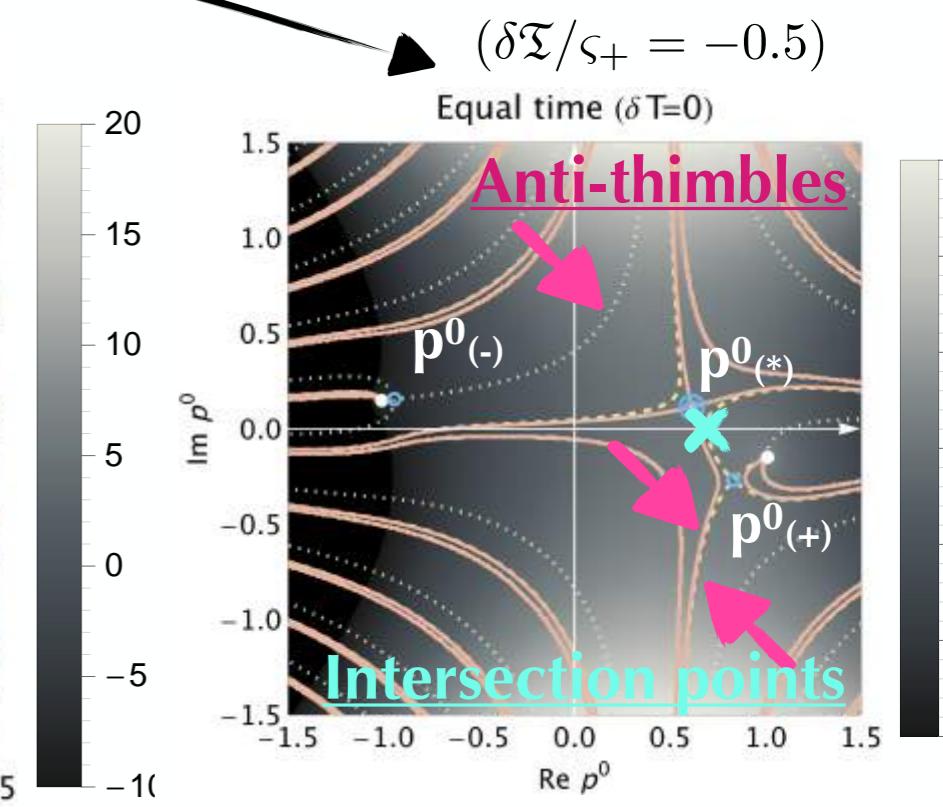
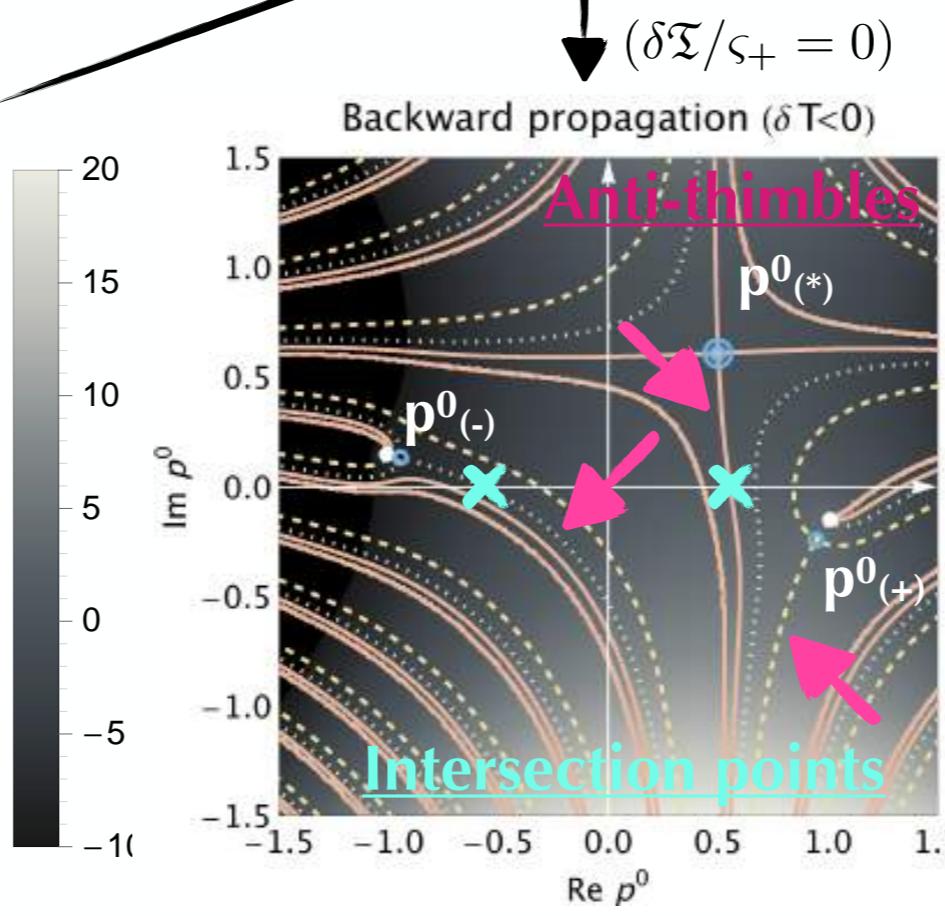
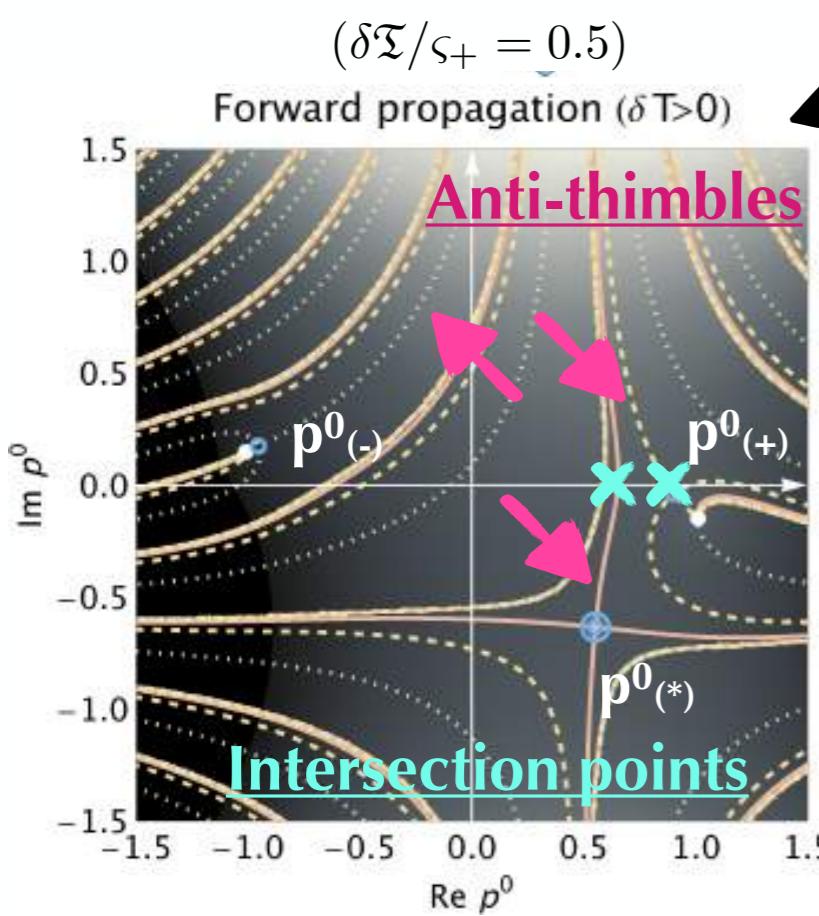
$$\frac{e}{\sqrt{2\pi}} \simeq 1.08$$

# Cross check of S(2→2): thimble decomposition



$$p_*^0(\mathbf{p}) = \omega_\varsigma(\mathbf{p}) - i \frac{\delta \mathfrak{T}}{\varsigma_+}$$

( $\varsigma_+ = 10$ ,  $\omega_\varsigma = 5$ ,  $\epsilon = 0.3$ , in the unit  $E_{\mathbf{p}} = 1$ )



Stokes phenomenon occurs.