2022 Saga-Yonsei Joint Workshop XVIII
Project: Exponentiated operators

Let us define a set of operators:

$$K_+ \equiv e^{2i\varphi_k} \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}^\dagger_{\boldsymbol{k}} \,,$$
$$K_- \equiv e^{-2i\varphi_k} \hat{a}_{-\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \,,$$
$$K_0 \equiv \frac{1}{2}\left(\hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{-\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} + 1 \right) ,$$

for some φ_k, where the following commutation relations are satisfied:

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}} \right] = 1 \,, \quad \text{otherwise zero} \,.$$

1. Show that K_+, K_- and K_0 close the algebra as

$$\left[K_-, K_+ \right] = 2K_0 \,,$$
$$\left[K_0, K_+ \right] = K_+ \,,$$
$$\left[K_0, K_- \right] = -K_- \,.$$

2. Show the the following relation is satisfied:

$$\exp\left(\alpha_0 K_0 + \alpha_+ K_+ + \alpha_- K_- \right) = e^{\gamma_+ K_+} e^{K_0 \log \gamma_0} e^{\gamma_- K_-} \,,$$

for constant α_0, α_+ and α_-, where

$$\gamma_0 \equiv \left(\cosh\theta - \frac{\alpha_0}{2\theta} \sinh\theta \right)^{-2} ,$$
$$\gamma_\pm \equiv \frac{2\alpha_\pm \sinh\theta}{2\theta\cosh\theta - \alpha_0 \sinh\theta} \,,$$
$$\theta^2 = \frac{\alpha_0^2}{4} - \alpha_+ \alpha_- \,.$$