QFT from Recursions

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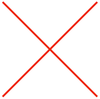
Feynman diagrams

- **In QFT course**

  Learn how to draw Feynman diagrams for computing amplitudes

Propagators:  $\frac{i}{p^2 - m^2 + i\epsilon}$

Interaction vertices:  $-i\lambda$

- simple rules, Lagrangian derivation, work for all theories.
- act as bookkeepers, visual tools to keep track of the allowed interactions of a particular mode
- It provide physical insight into the nature of particle interactions

- However, Feynman diagrams are impractical!!
- Cumbersome to figure out all possible topologies of diagrams and their symmetric factors.
- Number of Feynman diagrams grows exponentially.

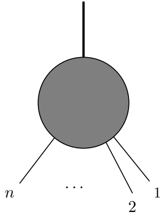
  For 2 to n tree-level gluon scattering amplitudes

$gg \Longrightarrow ng$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
Diagrams	$2,485$	$34,300$	$559,405$	$10,525,900$	$224,449,225$

- For seven gluons at one loop: $227,585$ Feynman diagrams.
- The expression for a single Feynman diagram can also be very complicated. For example, the three-graviton vertex has almost 100 terms
- However, the result of adding the contributions of many diagrams can be extraordinarily simple, when written in spinor-helicity variables – easy but stupid and inefficient expansion

- Two types of recursions for scattering amplitude: **On-shell vs Off-shell**
- **On-shell**: BCFW recursion relation [Britto, Cachazo, Feng, Witten ’04, ’05]
- It uses the analytic properties of tree-level scattering amplitudes via complex deformations of the external momenta without Feynman diagrams (**withtout Lagrangian**)
- It gives a recursion for on-shell scattering amplitude (all external legs are on-shell).
- The only input data: three-point amplitudes
- **Requirement**: amplitudes should vanish quickly at asymptotics (not applicalbe for scalar theories etc)
- Note that these recursions were intended mainly for tree-level scattering amplitudes

- Off-Shell recursion relation: recursion relation for off-shell currents [Berends, Giele '87]
- Off-Shell current for scalar theory $\Phi_{12\cdots n}$: Sum of all $(n+1)$-point Feynman graphs, where legs $1, 2, ..., n$ are on-shell gluons with amputations, and the $(n+1)$-leg is off-shell

$\Phi_{12\cdots n} =$ 

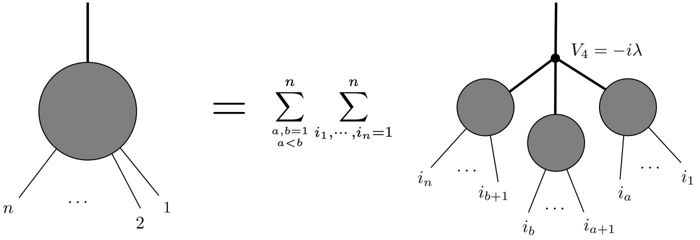
- Recall the LSZ reduction formula

$$\mathcal{A}(1,2,\cdots,n) = i^{n-1}(2\pi)^4\delta(k_1+\cdots+k_n)\tilde{G}_c(k_1,k_2,\cdots,k_n)\prod_{i=1}^{n}(k_i^2+m^2)$$

- $(n+1)$-point scattering amplitude from the off-shell current ($k_{ijk\cdots} = k_i + k_j + k_k + \cdots$)

$$\mathcal{A}(1,2,\cdots,n+1) = \lim_{-k_{1\cdots n}^2 \to m^2} \frac{i}{\hbar}(k_{12\cdots n}^2+m^2)\Phi_{12\cdots n}$$

- Based on recursive structure of tree-level Feynmann diagrams (Finite number of interaction vertices)



require off-shell currents with repeated momenta vanish: $\Phi_{i_1 \ldots j \ldots j \ldots i_n} = 0$

- Off-shell recursion relation for the off-shell currents

$$\Phi_{12\ldots n} = -\frac{\lambda}{3!}\frac{1}{k_{12\ldots n}^2 + m^2}\sum_{\substack{a,b=1\\a<b}}^{n}\sum_{i_1,i_n=1}^{n}\frac{1}{a!(b-a)!(n-b)!}\Phi_{i_1\ldots i_a}\Phi_{i_{a+1}\cdots i_b}\Phi_{i_{b+1}\cdots i_n}$$

- Initial condition: $\Phi_i = 1$

- Which one is more efficient? [Dinsdale, Ternick, Weinzierl '06]

  small number of legs: on-shell $>$ off-shell

  large number of legs: on-shell $<$ off-shell

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00005	0.00023	0.0009	0.003	0.011	0.030	0.09	0.27	0.7
BCFW	0.00001	0.00007	0.0003	0.001	0.006	0.037	0.19	0.97	5.5

- On-shell: Lagrangian is not necessary, not complitely general (standard model X)

  Off-shell: Lagrangian is crucial. SM is OK.,

“Conventional” Perturbiner Method

- We want to avoid the dependence on Feynmen vertices
- An alternative way to derive off-shell recursion relations systematically - useful for theories with complicated Feynman vertices, such as GR [Cho, Kim, KL ’21]
- Tree-level scattering amplitudes $\longleftrightarrow$ on-shell $\longleftrightarrow$ solutions of EoM
- How to relate the solutions of EoM and tree-level scattering amplitude?
- Perturbiner method connects solutions of classical EoM and tree-level scattering amplitude through the **off-shell currents** [Rosly, Selivanov ’96,’97], [Mizera,Skrzypek ’18]
- **Key ingredient**: perturbiner expansion, a generating function of the off-shell currents: using classical EoM instead of Feynman rules

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- Consider the following ansatz, so-called perturbiner expansion

$$\phi(x) = \sum_{i=1}^{N} \Phi_i e^{-ik_i\cdot x} + \sum_{i<j}^{N} \Phi_{ij} e^{-ik_{ij}\cdot x} + \sum_{i<j<k}^{N} \Phi_{ijk} e^{-ik_{ijk}\cdot x} + \cdots$$

  where k_i are on-shell, $k_i^2 + m^2 = 0$, and $k_{ijk\cdots} = k_i + k_j + k_k + \cdots$

- **Permutation invariance** (ex: $\Phi_{ijk} = \Phi_{jki} = \Phi_{kij} = \cdots$)

  use only plane waves $e^{-ik_{ijk\cdots}\cdot x}$ to separate the terms in the expansion

- A compact expression,

$$\phi(x) = \sum_{\mathcal{P}} \Phi_{\mathcal{P}} e^{-ik_{\mathcal{P}}\cdot x}$$

- $i, j, \cdots = 1, 2, \cdots, N$ are letters labeling single particles.

- $\mathcal{P}$ is an ordered word labeling the multi-particle states, $\mathcal{P} = ijkl\cdots$

  The length of the words their 'rank' and denoted as $|\mathcal{P}|, |\mathcal{Q}|, |\mathcal{R}|$ etc

- For non-commuting states (fermions, with color indices), we need unordered words

- Substitute the expansion into the classical EoM $(-\Box + m^2)\phi(x) = \frac{\lambda}{3!}\phi(x)^3$

$$\Phi_{\mathcal{P}} = -\frac{\lambda}{3!}\frac{1}{k_{\mathcal{P}}^2 + m^2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}}\Phi_{\mathcal{Q}}\Phi_{\mathcal{R}}\Phi_{\mathcal{S}}\,, \qquad |\mathcal{P}| > 1$$

- $\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}}$ means to sum over all possible distributions of the letters of the ordered words $\mathcal{P}$ into non-empty ordered words $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$.
- Ex: For $\mathcal{P} = 123$,

$$(\mathcal{Q}, \mathcal{R}, \mathcal{S}) \Longrightarrow \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$$

- Interestingly, this is the same exactly as the previous off-shell recursion relation
- Thus, the coefficients are the off-shell currents.

Let’s see a real example

Systematic derivation of the perturbiner expansion?

- How to derive the initial condition?
- How to generalise to loop levels?
- Is it possible to extend to other quantities other than scattering amplitudes?
- More general setup? (curved background generalisation)

Quantum Pertirbiner Expansion

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Consider ϕ^4-theory

1. Define quantum off-shell currents
2. Construct the quantum perturbiner expansion, a generating function of the quantum off-shell currents, by choosing the external source
3. looking for a quantum analogue of the classical EoM
4. Derive the recursion relation by substituting the quantum perturbiner expansion to the quantum EoM

$\Longrightarrow$ Employ the **quantum effective action formalism** (1PI diagrams generating functional)

- consider ϕ^4-theory with an external source $j_x := j(x)$

$$S[\phi, j] = \int \mathrm{d}^4x \Big[-\frac{1}{2}(\partial_\mu \phi(x))^2 - \frac{1}{2} m^2 \phi(x)^2 - \frac{\lambda}{4!}\phi(x)^4 + j(x)\phi(x) \Big] \,,$$

- The generating functional for connected diagrams, $W[j]$

$$e^{\frac{i}{\hbar} W[j]} = Z[j] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi, j]} \,.$$

- We denote the position and momentum space integrations

$$\int_{x,y\cdots} = \int \mathrm{d}^4x \, \mathrm{d}^4y \cdots \quad \text{and} \quad \int_{p,q,\cdots} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} \cdots$$

- The free propagator $D_{xy} := D(x,y)$, the inverse of the kinetic operator $K_{xy} = (-\Box + m^2)\delta^4(x-y)$,

$$\int_y K(x,y) D_{yz} = \delta^4(x-z) \,, \qquad D_{xy} = \int \frac{d^4p}{(2\pi)^4} \tilde{D}_p e^{ip\cdot(x-y)} \,,$$

where $\tilde{D}_p$ is the momentum space propagator,

$$\tilde{D}_p = \frac{1}{p^2 + m^2 - i\epsilon} \,.$$

- Expansion of $W[j]$ with respect to $j_x := j(x)$ around $j_x = 0$

$$W[j] = \sum_{n=2}^{\infty} \frac{1}{n!} \int_{x_1 x_2 \cdots x_n} \frac{\delta^n W[j]}{\delta j_{x_1} \delta j_{x_2} \cdots \delta j_{x_n}} \bigg|_{j=0} j_{x_1} j_{x_2} \cdots j_{x_n}$$

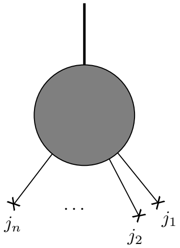
$$= -i\hbar \sum_{n=2}^{\infty} \frac{1}{n!} \int_{x_1 x_2 \cdots x_n} G_c(x_1, x_2, \cdots, x_n) \frac{ij_{x_1}}{\hbar} \frac{ij_{x_2}}{\hbar} \cdots \frac{ij_{x_n}}{\hbar} ,$$

- **The classical field** $\varphi_j(x)$: VEV of the scalar field $\phi(x)$ in the presence of j_x

$$\varphi_x := \frac{\delta W[j]}{\delta j_x} = \frac{\langle 0|\phi(x)|0\rangle_j}{\langle 0|0\rangle_j} , \qquad \varphi_x = \varphi_j(x)$$

$$\varphi_x = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{y_1, y_2, \cdots, y_n} G_c(x, y_1, y_2 \cdots, y_n) \frac{ij_{y_1}}{\hbar} \frac{ij_{y_2}}{\hbar} \cdots \frac{ij_{y_n}}{\hbar} .$$

$$\varphi_x = \sum_{n=1}^{\infty}$$



Quantum effective action

- Legendre transformation of $W[j]$ which interchanges the roles of j_x and φ_x

$$\Gamma[\varphi] = W[j] - \int_x j_x \varphi_x$$

- Variation of $\Gamma[\varphi]$ gives a field equation. At tree level, it reduces to the classical EoM

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi_x} = \int_y \left[\frac{\delta W[j]}{\delta j_y} \frac{\delta j_y}{\delta\varphi_x} - \frac{\delta j_y}{\delta\varphi_x}\varphi_y \right] - j_x = -j_x$$

$\Longrightarrow$ at least tree level, we may use φ_x for the perturbiner expansion

$$\phi(x) \Longrightarrow \varphi(x)$$

- We can compute $\Gamma[\varphi]$ from functional integration. However, the explicit form of $\Gamma[\varphi]$ is extremely complicated, and there is no closed form even in the one-loop level.
- **It is not suitable for deriving recursion relations**

- We further introduce the descendant fields $\psi_{x,y}$, $\psi'_{x,y,z}$, $\psi''_{x,y,z,w}$, $\cdots$ generated by acting multiple functional derivatives on φ_x:

$$\psi_{x,y} = \frac{\delta \varphi_x}{\delta j_y} = \frac{\delta^2 W[j]}{\delta j_x \delta j_y} ,$$

$$\psi'_{x,y,z} = \frac{\delta^2 \varphi_x}{\delta j_y \delta j_z} = \frac{\delta^3 W[j]}{\delta j_x \delta j_y \delta j_z} ,$$

$$\psi''_{x,y,z,w} = \frac{\delta^3 \varphi_x}{\delta j_y \delta j_z \delta j_w} = \frac{\delta^4 W[j]}{\delta j_x \delta j_y \delta j_z \delta j_w} .$$

- We may continue to arbitrarily higher-order descendant fields

$$\psi^{\overbrace{\prime\prime\cdots\prime}^{n}}_{x,x_1,x_2,\cdots,x_n} = \frac{\delta^n \varphi_x}{\delta j_{x_1} \delta j_{x_2} \cdots \delta j_{x_n}} = \frac{\delta^{n+1} W[j]}{\delta j_x \delta j_{x_1} \delta j_{x_2} \cdots \delta j_{x_n}} .$$

- Since the functional derivatives commute with each other, the ordering of the coordinates $x, x_1, \cdots$ is irrelevant

- $\hbar$-expansion = loop expansion
- We may expand φ_x and its descendants in $\hbar$,

$$\varphi_x = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \varphi_x^{(n)} , \qquad \psi_{x,y} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi_{x,y}^{(n)} , \qquad \psi'_{x,y,z} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi'^{(n)}_{x,y,z} .$$

- Let's go back to the expansion of $W[j]$

$$W[j] = -i\hbar \sum_{n=2}^{\infty} \frac{1}{n!} \int_{x_1 x_2 \cdots x_n} G_c(x_1, x_2, \cdots, x_n) \frac{ij_{x_1}}{\hbar} \frac{ij_{x_2}}{\hbar} \cdots \frac{ij_{x_n}}{\hbar} \, ,$$

- The external source j_x is completely arbitrary, and we are free to choose its form
- The dressed propagator $\mathbf{D}_{xy}$ is related to the exact two-point function,

$$\mathbf{D}_{xy} = \int_p e^{ip\cdot(x-y)} \tilde{\mathbf{D}}(p^2) = \frac{i}{\hbar} \langle 0|T\phi(x)\phi(y)|0\rangle \, .$$

$$\tilde{\mathbf{D}}(p^2) = \frac{1}{p^2 + m^2 - \Pi(p^2)} \, , \qquad \Pi(p^2) : \text{self-energy}$$

- Identify j_x to reproduce the LSZ reduction formula out of the expansion of $W[j]$

$$j_x = \sum_{i=1}^{N} \int_{y_i} \mathbf{K}_{xy_i} e^{-ik_i\cdot y_i} = \sum_{i=1}^{N} \tilde{\mathbf{K}}(-k_i) e^{-ik_i\cdot x}$$

$\mathbf{K}_{xy}$: the inverse of the dressed propagator $\mathbf{D}_{xy}$, satisfying $\int_y \mathbf{D}_{xy}\mathbf{K}_{yz} = \delta_{x,y}$ or $\tilde{\mathbf{K}}(p)\tilde{\mathbf{D}}(p) = 1$ in momentum space

- Substituting the j_x, $W[j]$ reduces to permutation sum of the LSZ reduction formula

$$\sum_{\text{Perm}[1,\cdots,N]} \mathcal{A}_{k_1,\cdots,k_N}$$

$$= -i\hbar(2\pi)^4\delta^4(k_{12\cdots n}) \sum_{i_1,i_2,\cdots,i_N} \tilde{G}_c(k_{i_1},k_{i_2},\cdots,k_{i_N})\left(\frac{i}{\hbar}\right)^N \prod_{i=1}^{N} \tilde{\mathbf{K}}(-k_i)$$

- Quantum off-shell current $\Phi_{i_1\cdots i_n}$: amputated correlation function with on-shell momenta k_i except for one off-shell leg which is assigned the momentum $k_{i_1\cdots i_n}$

$$\Phi_{i_1\cdots i_n} = \tilde{G}_c(-k_{i_1\cdots i_n}, k_{i_1}, \cdots, k_{i_n})\frac{i\tilde{\mathbf{K}}(-k_{i_1})}{\hbar}\cdots\frac{i\tilde{\mathbf{K}}(-k_{i_n})}{\hbar}\,.$$

- This is consistent with the previous definition of the off-shell current at tree level. Here $\tilde{G}_c(-k_{i_1\cdots i_n}, k_{i_1}, \cdots, k_{i_n})$ is the exact connected correlation function.

- This leads to our definition of the quantum perturbiner expansion of the classical field φ_x

$$\varphi_x = \sum_{i=1} \Phi_i e^{-ik_i\cdot x} + \sum_{i<j} \Phi_{ij} e^{-ik_{ij}\cdot x} + \cdots + \sum_{i_1<i_2<\cdots<i_n} \Phi_{i_1\cdots i_n} e^{-ik_{i_1\cdots i_n}\cdot x} + \cdots\,,$$

$$= \sum_{\mathcal{P}} \Phi_{\mathcal{P}} e^{-ik_{\mathcal{P}}\cdot x}\,,$$

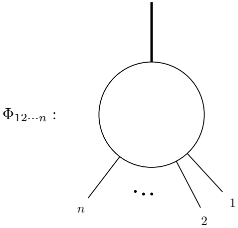
- The quantum perturbiner expansions for the descendant fields $\psi_{x,y}$ and $\psi'_{x,y,z}$

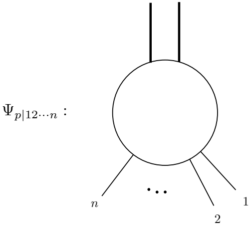
$$\psi_{x,y} = \int_p \Psi_{p|\emptyset} e^{ip\cdot(x-y)} + \sum_{\mathcal{P}} \int_p \Psi_{p|\mathcal{P}} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x} ,$$

$$\psi'_{x,y,z} = \sum_{\mathcal{P}} \int_{p,q} \Psi'_{p,q|\mathcal{P}} e^{ip\cdot(x-y)+iq\cdot(x-z)} e^{-ik_{\mathcal{P}}\cdot x} ,$$

- Descendant currents: $\Psi_{p|\mathcal{P}}$ and $\Psi'_{p,q|\mathcal{P}}$ are the quantum off-shell currents for the descendant fields.
- Here p and q are off-shell loop momenta, thus $\Psi_{p|\mathcal{P}}$ and $\Psi'_{p,q|\mathcal{P}}$ are intrinsic one-loop and two-loop quantities respectively.
- Note that the first descendant $\psi_{x,y}$ contains the zero-mode term.

- The graphical representation of the off-shell currents





- $\Phi_{12\cdots n}$ has an off-shell leg

  m-th descendant currents $\Psi''^{\cdots\prime}_{p_1 p_2 \cdots p_m|12\cdots n}$ have $m+1$ off-shell legs.

- Since $\tilde{G}_c(-k_{i_1\cdots i_n}, k_{i_1}, \cdots, k_{i_n})$ are expanded in powers of $\hbar$, the quantum off-shell currents are also expanded in $\hbar$

$$\Phi_{\mathcal{P}} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \Phi_{\mathcal{P}}^{(n)}, \quad \Psi_{p|\mathcal{P}} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \Psi_{p|\mathcal{P}}^{(n)}, \quad \psi'_{p,q|\mathcal{P}} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \Psi'^{(n)}_{p,q|\mathcal{P}},$$

- Scattering amplitude from the current

$$\mathcal{A}^{(n)}(k_1, \cdots, k_N) = \lim_{\substack{(k_{1\cdots N})^2 \\ \to -(m_{\text{phys}})^2}} \frac{i}{\hbar} \sum_{p=0}^{n} \tilde{\mathbf{K}}^{(p)}(k_{1\cdots N}) \Phi_{1\cdots N}^{(n-p)}.$$

  where $\tilde{\mathbf{K}}^{(p)}_{k_{1\cdots N}}$ is the inverse of the dressed propagator at p-loop order.

- It is convenient to define the amputated off-shell current $\hat{\Phi}_{1\cdots N}^{(n)}$ as

$$\hat{\Phi}_{1\cdots N}^{(n)} = \frac{i}{\hbar} \sum_{p=0}^{n} \tilde{\mathbf{K}}^{(p)}(k_{1\cdots N}) \Phi_{1\cdots N}^{(n-p)}.$$

Quantum Off-Shell Recursion Relation

- Recall that the off-shell recursion relation at tree-level is obtained from the classical EoM by substituting the perturbiner expansion
- Similarly, we shall construct the quantum off-shell recursion relation from the quantum perturbiner expansion we obtained
- We need to replace the classical EoM with its quantum counterpart
- Dyson–Schwinger (DS) equation as a quantum analogue of the classical EoM

- Several equivalent forms of DS eq. Here we represent in terms of φ_x
  [See QFT books by Ramnond or Brown]
- Identity for the total functional derivative within a functional integration

$$0 = \int \mathcal{D}\varphi_x \frac{\hbar}{i} \frac{\delta}{\delta \varphi_x} \, e^{\frac{i}{\hbar} S[\varphi, j]} = \int \mathcal{D}\varphi_x \frac{\delta S[\varphi, j]}{\delta \varphi_x} e^{\frac{i}{\hbar} S[\varphi, j]} \, .$$

- Denote the classical EoM as $\mathcal{F}[\varphi] = \frac{\delta S[\varphi, 0]}{\delta \varphi}$

$$\mathcal{F}\left(\frac{\hbar}{i} \frac{\delta}{\delta j_x} \right) Z[j] + j_x Z[j] = 0 \, , \qquad \varphi_x \iff \frac{\hbar}{i} \frac{\delta}{\delta j_x}$$

- Using this identiy $e^{-\frac{i}{\hbar} W[j]} \left(\frac{\hbar}{i} \frac{\delta}{\delta j_x} \right) e^{\frac{i}{\hbar} W[j]} = \varphi_x + \frac{\hbar}{i} \frac{\delta}{\delta j_x}$,

$$\mathcal{F}\left(\varphi_x + \frac{\hbar}{i} \frac{\delta}{\delta j_x} \right) + j_x = 0$$

- Thus the DS equation is a deformation of the classical EoM by $\varphi_x \to \varphi_x + \frac{\hbar}{i} \frac{\delta}{\delta j_x}$
- A quantization scheme?

- DS equation for ϕ^4-theory:

$$\int_y K(x,y)\varphi_y + \frac{\lambda}{3!}\varphi_x^3 = j_x - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_x\frac{\delta\varphi_x}{\delta j_x} + \hbar^2\frac{\lambda}{3!}\frac{\delta^2\varphi_x}{\delta j_x\delta j_x}\,.$$

- How to solve it? Recall the descendant fields!

$$\psi_{x,y} = \frac{\delta\varphi_x}{\delta j_y}\,, \qquad \psi'_{x,y,z} = \frac{\delta^2\varphi_x}{\delta j_y\delta j_z}$$

- Strategy: Treat the functional derivatives for φ_x as new independent field variables

$$\varphi_x = \int_y D_{xy}\left(j_y - \frac{\lambda}{3!}\varphi_y^3\right) + i\hbar\frac{\lambda}{2}\int_y D_{xy}\varphi_y\psi_{y,y} + \hbar^2\frac{\lambda}{3!}\int_y D_{xy}\psi'_{y,y,y}$$

where $\psi_{x,x} = \lim_{y\to x}\psi_{x,y}$, and $\psi'_{x,x,x} = \lim_{\substack{y\to x\\ z\to x}}\psi'_{x,y,z}$

- It is not sufficient to solve the equation: # of eq $<$ # of variables

- Descendant equations: acting $\frac{\delta}{\delta j_x}$ multiple times on the DS equation

$$\psi_{x,z} = D_{xz} - \frac{\lambda}{2}\int_y D_{xy}\phi_y^2\psi_{y,z} + i\hbar\frac{\lambda}{2}\int_y D_{xy}\left(\phi_y\psi'_{y,y,z} + \psi_{y,z}\psi_{y,y}\right)$$
$$+ \hbar^2\frac{\lambda}{3!}\int_y D_{xy}\psi''_{y,y,y,z}\,,$$
$$\psi'_{x,z,w} = -\frac{\lambda}{2}\int_y D_{xy}\Big(2\phi_y\psi_{y,w}\psi_{y,z} + \phi_y^2\psi'_{y,z,w}\Big)$$
$$+ i\hbar\frac{\lambda}{2}\int_y D_{xy}\Big(\psi_{y,w}\psi'_{y,y,z} + \phi_y\psi''_{y,y,z,w} + \psi'_{y,z,w}\psi_{y,y} + \psi_{y,z}\psi'_{y,y,w}\Big)$$
$$+ \hbar^2\frac{\lambda}{3!}\int_y D_{xy}\psi'''_{y,y,y,z,w}\,.$$

- However, we encounter new descendant fields, $\psi''_{x,z}$ and $\psi'''_{x,z,w}$
- Do we have to repeat this procedure forever??

$\hbar$-expansion of the field equations

- Substitute the $\hbar$-expansion of all the fields and keep the terms at a fixed order in $\hbar$

$$\varphi_x = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \varphi_x^{(n)} \,, \qquad \psi_{x,y} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi_{x,y}^{(n)} \,, \qquad \psi'_{x,y,z} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi'^{(n)}_{x,y,z} \,.$$

- We can truncate the new descendant fields because these are higher $\hbar$-order terms in general.
- Once we have the DS equation and its descendants at a specific order in $\hbar$, the recursion relation for the quantum off-shell currents can be derived by substituting the quantum perturbiner expansions
- We will construct the quantum off-shell recursion relation up to two-loop level and solve the one-loop off-shell currents.

- The tree-level DS equation is the same as the classical EoM without descendants

$$\varphi_x^{(0)} = \int_y D_{xy} \left(j_y^{(0)} - \frac{\lambda}{3!} \left(\varphi_y^{(0)} \right)^3 \right) .$$

- Substitute the following perturbiner expansion at $\hbar^0$-order

$$\varphi_x^{(0)} = \sum_{\mathcal{P}} \Phi_{\mathcal{P}}^{(0)} e^{-ik_{\mathcal{P}} \cdot x}$$

- We reproduce the conventional tree-level off-shell recursion relation

$$\Phi_{\mathcal{P}}^{(0)} = -\frac{\lambda}{3!} \frac{1}{k_{\mathcal{P}}^2 + m^2} \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi_{\mathcal{Q}}^{(0)} \Phi_{\mathcal{R}}^{(0)} \Phi_{\mathcal{S}}^{(0)} , \qquad |\mathcal{P}| > 1 ,$$

- Initial condition: from the rank-1

$$\sum_i \Phi_i^{(0)} e^{-k_i \cdot x} = \int_y D_{xy} j^{(0)}(y) = \sum_{i=1}^N \int_{y,p} \frac{e^{ip\cdot(x-y)}}{p^2 + m^2} (k_i^2 + m^2) e^{-ik_i \cdot y} .$$

and we have $\Phi_i^{(0)} = 1$.

- Substituting the $\hbar$-expansion of φ_x and $\psi_{x,x}$ into the DS eq and keeping $(\hbar)^1$-order terms

$$\phi_x^{(1)} = \int_y D_{xy}\left[j_y^{(1)} - \frac{\lambda}{2}\left(\left(\phi_y^{(0)}\right)^2 \phi_y^{(1)} + \phi_y^{(0)} \psi_{y,y}^{(0)}\right)\right].$$

- We need a field equation for $\psi_{y,y}^{(0)}$. Keeping the $\hbar^0$-order terms in the first descendant equation

$$\psi_{x,z}^{(0)} = D_{xz} - \frac{\lambda}{2}\int_y D_{xy}\left(\phi_y^{(0)}\right)^2 \psi_{y,z}^{(0)}.$$

- We have a pair of independent equations for the two unknown fields at one-loop.

- The perturbiner expansion for $\varphi_x^{(1)}$ and $\psi_{p|\mathcal{P}}^{(0)}$ are

$$\varphi_x^{(1)} = \sum_{\mathcal{P}} \Phi_{\mathcal{P}}^{(1)} e^{-ik_{\mathcal{P}}\cdot x}\,,$$

$$\psi_{x,y}^{(0)} = \int_p \Psi_{p|\emptyset}^{(0)} e^{ip\cdot(x-y)} + \sum_{\mathcal{P}} \int_p \Psi_{p|\mathcal{P}}^{(0)} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x}\,.$$

- Substituting the perturbiner expansions into the pair of equations, we obtain the recursion relations at one-loop level

$$\Phi_{\mathcal{P}}^{(1)} = -\frac{\lambda}{2}\frac{1}{(k_{\mathcal{P}})^2+m^2}\left(\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(0)}\Phi_{\mathcal{S}}^{(1)} + \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}} \int_p \Phi_{\mathcal{Q}}^{(0)}\Psi_{p|\mathcal{R}}^{(0)}\right),$$

$$\Psi_{p|\mathcal{P}}^{(0)} = \frac{\lambda}{2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \frac{1}{(p-k_{\mathcal{P}})^2+m^2}\Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(0)}\Psi_{p|\mathcal{S}}^{(0)}$$

- Note that D_{xz} on the right-hand side of the first descendant equation does not depend on the external momenta k_i
- It corresponds to the zero-mode, $\Psi^{(0)}_{p|\emptyset}$ and we can determine $\Psi^{(0)}_{p|\emptyset}$

$$\Psi^{(0)}_{p|\emptyset} = \frac{1}{p^2+m^2}\,,$$

- We can determine the rank-one current $\Phi^{(1)}_i$ from the DS equation at one-loop

$$\begin{aligned}\sum_i \Phi^{(1)}_i e^{-ik_i\cdot x} &= \int_y D_{xy}\left[j^{(1)}_y - \frac{\lambda}{2}\sum_i \Phi^{(0)}_i e^{-k_i\cdot x}\int_p \Psi^{(0)}_{p|\emptyset}\right] \\ &= \frac{i}{\hbar}\frac{\lambda}{2}\sum_i \int_y D_{xy}\left[\int_p \frac{1}{p^2+m^2}e^{-ik_i\cdot y} - \int_p \frac{1}{p^2+m^2}e^{-ik_i\cdot y}\right] \\ &= 0\,.\end{aligned}$$

- Thus the initial condition for the one-loop off-shell current is trivial,

$$\Phi^{(1)}_i = 0$$

Expansion of DS equaiton: Two loop

- Consider the DS equation at second order in $\hbar$,

$$\varphi_x^{(2)} = \int_y D_{xy} j_x^{(2)} - \frac{\lambda}{2} \int_y D_{xy} \left((\varphi_y^{(0)})^2 \varphi_y^{(2)} + 2\varphi_y^{(0)} (\varphi_y^{(1)})^2 + (\varphi_y^{(0)} \psi_{y,y}^{(1)} + \varphi_y^{(1)} \psi_{y,y}^{(0)}) + \frac{1}{3} \psi'^{(0)}_{y,y,y} \right)$$

- The first descendant equation at one-loop order,

$$\psi_{x,z}^{(1)} = -\frac{\lambda}{2} \int_y D_{xy} \left(2\phi_y^{(0)} \phi_y^{(1)} \psi_{y,z}^{(0)} + (\phi_y^{(0)})^2 \psi_{y,z}^{(1)} \right) - \frac{\lambda}{2} \int_y D_{xy} \left(\phi_y^{(0)} \psi'^{(0)}_{y,y,z} + \psi_{y,y}^{(0)} \psi_{y,z}^{(0)} \right)$$

- There is a new descendant field $\psi'^{(0)}_{x,y,z}$ satisfying the second descendant equation

$$\psi'^{(0)}_{x,z,w} = -\frac{\lambda}{2} \int_y D_{xy} \left(2\phi_y^{(0)} \psi_{y,z}^{(0)} \psi_{y,w}^{(0)} + (\phi_y^{(0)})^2 \psi'^{(0)}_{y,z,w} \right)$$

- The quantum perturbiner expansion for the fields in the two-loop DS equation is

$$
\begin{aligned}
\varphi_x^{(2)} &= \sum_{\mathcal{P}} \Phi_{\mathcal{P}}^{(2)} e^{-ik_{\mathcal{P}}\cdot x}\,, \\
\psi_{x,y}^{(1)} &= \int_p \Psi_{p|\emptyset}^{(1)} e^{ip\cdot(x-y)} + \sum_{\mathcal{P}} \int_p \Psi_{p|\mathcal{P}}^{(1)} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x}\,, \\
\psi_{x,y,z}^{\prime(0)} &= \sum_{\mathcal{P}} \int_{p,q} \Psi_{p,q|\mathcal{P}}^{(0)} e^{ip\cdot(x-y)+iq\cdot(x-z)} e^{-ik_{\mathcal{P}}\cdot x}\,.
\end{aligned}
$$

- Substituting the perturbiner expansion into the two-loop DS equation and its descendants,

$$
\begin{aligned}
\Phi_{\mathcal{P}}^{(2)} &= -\frac{\lambda}{2}\frac{1}{k_{\mathcal{P}}^2+m^2}\Bigg(\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \left(\Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(0)}\Phi_{\mathcal{S}}^{(2)} + 2\Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(1)}\Phi_{\mathcal{S}}^{(1)}\right) \\
&\qquad + \sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}} \int_p \left(\Phi_{\mathcal{Q}}^{(0)}\Psi_{p|\mathcal{R}}^{(1)} + \Phi_{\mathcal{Q}}^{(1)}\Psi_{p|\mathcal{R}}^{(0)}\right) + \frac{1}{3}\int_{p,q} \Psi_{p,q|\mathcal{P}}^{\prime(0)}\Bigg)\,, \\
\Psi_{p|\mathcal{P}}^{(1)} &= -\frac{\lambda}{2}\frac{1}{(p-k_{\mathcal{P}})^2+m^2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \left(2\Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(1)}\Psi_{p|\mathcal{S}}^{(0)} + \Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(0)}\Psi_{p|\mathcal{S}}^{(1)}\right) \\
&\quad - \frac{\lambda}{2}\frac{1}{(p-k_{\mathcal{P}})^2+m^2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}} \int_q \left(\Phi_{\mathcal{Q}}^{(0)}\Psi_{p,q|\mathcal{R}}^{\prime(0)} + \Psi_{p|\mathcal{Q}}^{(0)}\Psi_{q|\mathcal{R}}^{(0)}\right)\,, \\
\Psi_{p,q|\mathcal{P}}^{\prime(0)} &= -\frac{\lambda}{2}\frac{1}{(p+q-k_{\mathcal{P}})^2+m^2}\sum_{\mathcal{P}=\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}} \left(2\Phi_{\mathcal{Q}}^{(0)}\Psi_{p|\mathcal{R}}^{(0)}\Psi_{q|\mathcal{S}}^{(0)} + \Phi_{\mathcal{Q}}^{(0)}\Phi_{\mathcal{R}}^{(0)}\Psi_{p,q|\mathcal{S}}^{\prime(0)}\right)\,.
\end{aligned}
$$

- The rank-1 current $\Phi_i^{(2)}$ satisfies

$$\sum_i \Phi_i^{(2)} e^{-ik_i\cdot x} = \int_y D_{xy} j_y^{(2)} - \frac{\lambda}{2}\int_y \sum_i D_{xy}\left(\Phi_i^{(0)}\int_p \Psi_{p|\emptyset}^{(1)} + \frac{1}{3}\int_{p,q}\Psi'^{(0)}_{p,q|i}\right)e^{-k_i\cdot y}\,.$$

- $\Psi_{p|\emptyset}^{(1)}$ satisfies the zero-mode sector of the first descendant of the DS equation,

$$\Psi_{p|\emptyset}^{(1)} = -\frac{\lambda}{2}\frac{1}{p^2+m^2}\int_q \Psi_{p|\emptyset}^{(0)}\Psi_{q|\emptyset}^{(0)}\,, = -\frac{\lambda}{2}\left(\frac{1}{p^2+m^2}\right)^2\int_q \frac{1}{q^2+m^2}\,.$$

- The lowest rank of the second descendant field is $\Psi'^{(0)}_{p,q|i}$ satisfying

$$\begin{aligned}\Psi'^{(0)}_{p,q|i} &= -\frac{\lambda}{2}\frac{2}{(p+q-k_i)^2+m^2}\Phi_i^{(0)}\Psi_{p|\emptyset}^{(0)}\Psi_{q|\emptyset}^{(0)}\\ &= -\frac{\lambda}{2}\frac{2}{(p+q-k_i)^2+m^2}\frac{1}{p^2+m^2}\frac{1}{q^2+m^2}\,.\end{aligned}$$

- Combining all these, we can show the initial condition for $\Phi_i^{(2)}$ is also trivial,

$$\Phi_i^{(2)} = 0\,.$$

- We may expect that this is a general property of the loop-level currents,

$$\Phi_i^{(n)} = 0\,, \qquad \text{for } n > 1\,.$$

Solving the recursion relation
at one loop

General structure

- The recursion relation provides the integrand of the loop integral
- We need to perform the loop integration separately
- Regularization and renormalization is required to get finite quantities
- It is easy to implement the recursions in computer programs (mathematica)

- rank-one current is the initial condition : $\Phi_i^{(1)} = 0$, $\Phi_i^{(0)} = 1$,
- one-loop rank-2 current $\Phi_{ij}^{(1)}$ satisfies

$$\Phi_{ij}^{(1)} = -\frac{\lambda}{2} \frac{1}{k_{ij}^2 + m^2} \sum_{i,j} \left(\Phi_{ij}^{(0)} \int_p \Psi_{p|\emptyset}^{(0)} + 2\Phi_i^{(0)} \int_p \Psi_{p|j}^{(0)} \right).$$

- It is straightforward to show that $\Psi_{p|i}^{(0)} = 0$ using the recursion relations, and we have

$$\Phi_{ij}^{(1)} = 0\,.$$

- This result is consistent with the fact that there is no one-loop three-point function in ϕ^4-theory.

Rank three

- The one-loop rank-3 current from the recursion relation

$$\Phi^{(1)}_{ijk} = -\frac{\lambda}{2}\frac{1}{(k_{ijk})^2+m^2}\sum_{\text{Perm}[i,j,k]}\left(\frac{1}{2}\int_p \Phi^{(0)}_i \Psi^{(0)}_{p|jk} + \frac{1}{3!}\int_p \Phi^{(0)}_{ijk}\Psi^{(0)}_{p|\emptyset}\right)$$

- We need to determine $\Phi^{(0)}_{ijk}$ and $\Psi^{(0)}_{p|ij}$. From the tree-level recursion relation

$$\Phi^{(0)}_{ijk} = -\frac{\lambda}{3!}\frac{1}{k^2_{ijk}+m^2}\sum_{\text{Perm}[i,j,k]}\Phi^{(0)}_i\Phi^{(0)}_j\Phi^{(0)}_k = -\frac{\lambda}{k^2_{ijk}+m^2}\,.$$

- Next, we consider $\Psi^{(0)}_{p|ij}$

$$\begin{aligned}\Psi^{(0)}_{p|ij} &= -\lambda\frac{1}{(p-k_{ij})^2+m^2}\Phi^{(0)}_i\Phi^{(0)}_j\Psi^{(0)}_{p|\emptyset}\\ &= -\lambda\frac{1}{(p-k_{ij})^2+m^2}\frac{1}{p^2+m^2}\,.\end{aligned}$$

- If we collect all the ingredients, the one-loop rank-3 current reduces to

$$\begin{aligned}\Phi^{(1)}_{ijk} = \frac{\lambda^2}{2}\frac{1}{k^2_{ijk}+m^2}\Bigg(&\frac{1}{k^2_{ijk}+m^2}\int_p\frac{1}{p^2+m^2}\\ &+\int_p\frac{1}{(p^2+m^2)\left((p-k_{ij})^2+m^2\right)} + (ij\to jk\to ik)\Bigg)\,.\end{aligned}$$

- The amputated off-shell current at one loop order is

$$\hat{\Phi}^{(1)}_{ijk} = \frac{i}{\hbar}\Big(\tilde{\mathbf{K}}^{(1)}(k_{ijk})\Phi^{(0)}_{ijk} + \tilde{\mathbf{K}}^{(0)}(k_{ijk})\Phi^{(1)}_{ijk}\Big).$$

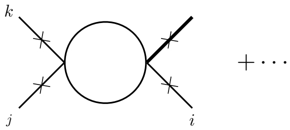
where

$$\tilde{\mathbf{K}}^{(1)}(k_{ijk}) = -\Pi^{(1)} = -\frac{\hbar}{i}\frac{\lambda}{2}\int_p \frac{1}{p^2+m^2}\,, \qquad \tilde{\mathbf{K}}^{(0)}(k_{ijk}) = k^2_{ijk} + m^2$$

- The corresponding amputated off-shell current $\hat{\Phi}^{(1)}_{ijk}$ is given by

$$\hat{\Phi}^{(1)}_{ijk} = \frac{\lambda^2}{2}\int_p \frac{1}{\big(p^2+m^2\big)\big((p-k_{ij})^2+m^2\big)} + \big(ij \to jk \to ik\big)\,.$$

- The diagrammatical representation of the rank-3 amputated current



Rank four

- From the recursion relation, $\Phi^{(1)}_{ijkl}$ is given by

$$\Phi^{(1)}_{ijkl} = -\frac{\lambda}{2}\frac{1}{k^2_{ijkl}+m^2}\sum_{\text{Perm}[ijkl]}\Bigg[\frac{1}{2}\Phi^{(0)}_i\Phi^{(0)}_j\Phi^{(1)}_{kl} + \Phi^{(0)}_{ij}\Phi^{(0)}_k\Phi^{(1)}_l + \int_p\left(\frac{1}{4!}\Phi^{(0)}_{ijkl}\Psi^{(0)}_{p|\emptyset} + \frac{1}{3!}\Phi^{(0)}_{ijk}\Psi^{(0)}_{p|l} + \frac{2}{(2!)^2}\Phi^{(0)}_{ij}\Psi^{(0)}_{p|kl} + \frac{1}{3!}\Phi^{(0)}_i\Psi^{(0)}_{p|jkl}\right)\Bigg]$$

- Using the previous results, $\Phi^{(0)}_{ij} = \Phi^{(0)}_{ijkl} = \Phi^{(1)}_{ij} = \Psi^{(0)}_{p|i} = 0$, and the remaining unknown current is $\Psi^{(0)}_{p|ijk}$. From the recursion relation, it is given by

$$\Psi^{(0)}_{p|ijk} = \frac{\lambda}{2}\frac{1}{(p-k_{ijk})^2+m^2}\sum_{\text{perm}[ijk]}\Phi^{(0)}_i\Phi^{(0)}_j\Psi^{(0)}_{p|k} = 0\,.$$

- Since $\Psi^{(0)}_{p|i} = 0$, $\Psi^{(0)}_{p|ijk}$ also vanishes. Thus, this shows that the one-loop rank-4 current $\Phi^{(1)}_{ijkl}$ and the one-loop 5-point amplitude vanish,

$$\Phi^{(1)}_{ijkl} = 0$$

- The one-loop rank-5 current $\Phi^{(1)}_{i_1 i_2 \cdots i_5}$ satisfies the recursion relation

$$\Phi^{(1)}_{i_1\cdots i_5} = -\frac{\lambda}{2}\frac{1}{k^2_{i_1\cdots i_5}+m^2}\sum_{\text{Perm}[i_1,\cdots,i_5]}\Bigg[\frac{1}{3!}\Phi^{(0)}_{i_1}\Phi^{(0)}_{i_2}\Phi^{(1)}_{i_3 i_4 i_5} + \int_p \left(\frac{1}{5!}\Phi^{(0)}_{i_1\cdots i_5}\Psi^{(0)}_{p|\emptyset} + \frac{1}{2!}\frac{1}{3!}\Phi^{(0)}_{i_1 i_2 i_3}\Psi^{(0)}_{p|i_4 i_5} + \frac{1}{4!}\Phi^{(0)}_{i_1}\Psi^{(0)}_{p|i_2 i_3 i_4 i_5}\right)\Bigg]$$

- We have otained $\Phi^{(1)}_{ijk}$, $\Phi^{(0)}_{i_1\cdots i_5}$, $\Psi^{(0)}_{p|\emptyset}$ and $\Psi^{(0)}_{p|ij}$. The only unknown current is $\Psi^{(0)}_{p|ijkl}$.
- The rank-4 descendant current $\Psi^{(0)}_{p|ijkl}$ can be obtained from the recursion relation

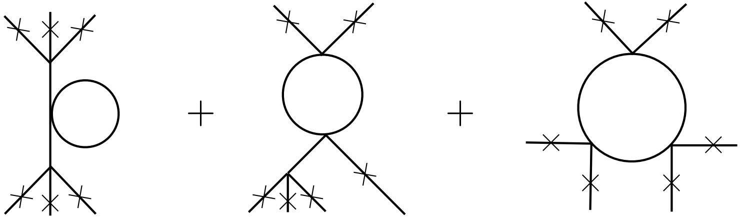
$$\Psi^{(0)}_{p|ijkl} = \frac{\lambda}{2}\frac{1}{(p-k_{ijkl})^2+m^2}\sum_{\text{Perm}[ijkl]}\left(\frac{1}{2!}\Phi^{(0)}_i\Phi^{(0)}_j\Psi^{(0)}_{p|kl} + \frac{1}{3!}\Phi^{(0)}_i\Phi^{(0)}_{jkl}\Psi^{(0)}_{p|\emptyset}\right)$$

$$= -\frac{\lambda^2}{4}\frac{1}{(p-k_{ijkl})^2+m^2}\sum_{\text{Perm}[ijkl]}\frac{1}{p^2+m^2}\left[\frac{1}{(p-k_{kl})^2+m^2} + \frac{1}{3}\frac{1}{k^2_{jkl}+m^2}\right]$$

- The amputated off-shell current at this order $\hat{\Phi}^{(1)}_{i_1 \cdots i_5}$ is given by

$$\hat{\Phi}^{(1)}_{i_1 \cdots i_5} = -\frac{\lambda}{2} \sum_{\text{Perm}[i_1, \cdots, i_5]} \left[\frac{1}{3!} \Phi^{(1)}_{i_3 i_4 i_5} + \int_p \left(\frac{1}{2 \cdot 3!} \Phi^{(0)}_{i_1 i_2 i_3} \Psi^{(0)}_{p|i_4 i_5} + \frac{1}{4!} \Psi^{(0)}_{p|i_2 i_3 i_4 i_5} \right) \right].$$

- We can represent the rank-5 current diagrammatically



$$\sum_{\text{Perm}[i_1, \cdots, i_5]} \left[\frac{1}{3!} \Phi^{(1)}_{i_3 i_4 i_5} + \int_p \frac{1}{2 \cdot 3!} \Phi^{(0)}_{i_1 i_2 i_3} \Psi^{(0)}_{p|i_4 i_5} \right] \qquad \sum_{\text{Perm}[i_1, \cdots, i_5]} \frac{1}{4!} \int_p \Psi^{(0)}_{p|i_2 i_3 i_4 i_5}$$

It precisely reproduces the one-loop 6-point scattering amplitude.

Recursion relations for correlation functions

- Recall that we have chosen the extenral source to reproduce the LSZ reduction formula out of the expansion of $W[j]$
- However, the choice of the external source is not unique – we may replace it depending on our purpose.
- The off-shell current describes different quantities as we modify the external source.
- We replace the external source $j(x)$ with a new current $\check{j}(x)$,

$$\check{j}(x) = \frac{\hbar}{i} \sum_{i=1}^{N} \int_{y_i} \delta^4(x - y_i) e^{-ik_i \cdot y_i} = \frac{\hbar}{i} \sum_{i=1}^{N} e^{-ik_i \cdot x} ,$$

  where k_i are the external momenta without an on-shell condition.
- We have dropped the inverse propagator $\mathbf{K}_{xy}$ in the external source because we do not have to amputate the external legs when computing correlation functions
- Most of the results up to now still hold!

- The classical field $\check{\varphi}_x$ with respect to the new external source is given by

$$\check{\varphi}_x = \frac{\delta W[\check{j}]}{\delta \check{j}(x)} ,$$

- the classical field is expanded as

$$\begin{aligned} \check{\varphi}_x &= \sum_{i=1} \check{\Phi}_i e^{-ik_i \cdot x} + \sum_{i<j} \check{\Phi}_{ij} e^{-ik_{ij} \cdot x} + \cdots + \sum_{i_1 < i_2 < \cdots < i_N} \check{\Phi}_{i_1 \cdots i_N} e^{-ik_{i_1 \cdots i_N} \cdot x} \\ &= \sum_{\mathcal{P}} \check{\Phi}_{\mathcal{P}} e^{-ik_{\mathcal{P}} \cdot x} . \end{aligned}$$

- Then the rank-n off-shell currents in the perturbiner expansion of φ_x provide the momentum space $(n+1)$-point correlation functions

$$\check{\Phi}_{i_1 i_2 \cdots i_n} = \tilde{G}_c(-k_{i_1 i_2 \cdots i_n}, k_{i_1}, k_{i_2}, \cdots , k_{i_n}) .$$

- The off-shell currents are identified with correlation functions directly.

- Since the form of the perturbiner expansion and the DS equation are invariant with respect to the choice of external source, the off-shell recursion relations are the same as the previous results
- The only difference is the initial condition because it directly depends on the choice of external source explicitly.
- We can determine the initial condition similar way
- Solving the recursions is straightforward

Quantum Perturbiner Method for Yang–Mills theory

- Most of the structure is almost parallel with the previous case.
- However, there is a crucial difference: the gauge symmetry.
- We investigate how to treat the gauge symmetry in the framework of the quantum perturbiner method. We follow the conventional Faddeev–Popov (FP) ghost method in QFT.
- We impose a gauge choice and introduce ghosts to remove the unphysical gauge redundancies. We also present the perturbiner expansion for the entire ghost sector including descendants of the ghosts.

- We introduce classical fields $\mathcal{A}_j^{a\mu}(x)$, $\mathcal{C}_j^a(x)$ and $\bar{\mathcal{C}}_j^a(x)$ which are the VEVs of A_μ^a, c^a and $\bar{c}^a$ in the presence of their external sources $j_x^{a\mu}$, η_x^a and $\bar{\eta}_x^a$

$$\mathcal{A}_x^{a\mu} = \frac{\delta W[j,\eta,\bar{\eta}]}{\delta j_x^{a\mu}}\,, \qquad \mathcal{C}_x^a = \frac{\delta_L W[j,\eta,\bar{\eta}]}{\delta \bar{\eta}_x^a}\,, \qquad \bar{\mathcal{C}}_x^a = \frac{\delta_R W[j,\eta,\bar{\eta}]}{\delta \eta_x^a}\,,$$

- Descendant fields

$$\psi_{x,y}^{a\mu,b\nu} = \frac{\delta A_x^{a\mu}}{\delta j_y^{b\nu}}\,, \qquad \psi'^{\,a\mu,b\nu,c\rho}_{x,y,z} = \frac{\delta^2 A_x^{a\mu}}{\delta j_y^{b\nu}\delta j_z^{c\rho}}\,,$$

$$\gamma_{x,y}^{a,b} = \frac{\delta_L c_x^a}{\delta \eta_x^b} = \frac{\delta_R c_x^a}{\delta \bar{\eta}_x^b}\,, \quad \theta_{x,y}^{a,b\mu} = \frac{\delta c_x^a}{\delta j_y^{b\mu}}\,, \quad \bar{\theta}_{x,y}^{a,b\mu} = \frac{\delta \bar{c}_x^a}{\delta j_y^{b\mu}}$$

- We now assign external sources for gluons, $j_x^{a\mu} := j^{a\mu}(x)$, and ghost fields, $\eta_x^a = \eta^a(x)$ and $\bar{\eta}_x^a := \bar{\eta}^a(x)$, as follows:

$$j_x^{a\mu} = \sum_{i=1}^{N} \int_{y_i} \mathbf{K}_{\mu,\nu_i}^{a,b_i}(x,y_i)\epsilon_i^{\nu_i} e^{-ik_i\cdot y_i} = \sum_{i=1}^{N} \tilde{\mathbf{K}}_{\mu,\nu_i}^{a,b_i}(-k_i)\epsilon_i^{\nu_i} e^{-ik_i\cdot x}\,,$$

$$\eta_x^a = 0\,, \qquad \bar{\eta}_x^a = 0\,,$$

- Quantum perturbiner expansion

$$\mathcal{A}^a_\mu = \sum_{\mathcal{P}} J^{a\mu}_{\mathcal{P}} e^{-ik_{\mathcal{P}}\cdot x}\,, \qquad \mathcal{C}^a_x = \bar{\mathcal{C}}^a_x = 0$$

- We also introduce the quantum perturbiner expansion for the descendant fields,

$$\psi^{a\mu,b\nu}_{x,y} = \Psi^{a\mu,b\nu}_{p|\emptyset} e^{ip\cdot(x-y)} + \sum_{\mathcal{P}} \int_p \Psi^{a\mu,b\nu}_{p|\mathcal{P}} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x}\,,$$

$$\psi'^{a\mu,b\nu,c\rho}_{x,y,z} = \sum_{\mathcal{P}} \int_{p,q} \Psi^{a\mu,b\nu,c\rho}_{p,q|\mathcal{P}} e^{ip\cdot(x-y)+iq\cdot(x-z)} e^{-ik_{\mathcal{P}}\cdot x}\,,$$

$$\gamma^{a,b}_{x,y} = \sum_{\mathcal{P}} \int_p \Gamma^{a,b}_{p|\mathcal{P}} e^{ip\cdot(x-y)} e^{-ik_{\mathcal{P}}\cdot x}\,,$$

- we can derive the DS equation for pure YM theory from the classical EoM by deforming $\mathcal{A}_x^{a\mu} \to \mathcal{A}_x^{a\mu} + \frac{\hbar}{i}\frac{\delta}{\delta j_x^{a\mu}}, \mathcal{C}^a \to \mathcal{C}^a + \frac{\hbar}{i}\frac{\delta_L}{\delta \bar{\eta}_x^a}$ and $\bar{\mathcal{C}}^a \to \bar{\mathcal{C}}^a + \frac{\hbar}{i}\frac{\delta_R}{\delta \eta_x^a}$.

$$
\begin{aligned}
\int_y K_{x,y}\mathcal{A}_y^{a\mu} = {} & j_x^{a\mu} + g f_{abc}\Big(\mathcal{A}_x^{b\nu}(2\partial_\nu \mathcal{A}_x^{c\mu} - \partial_\mu \mathcal{A}_x^{c\nu}) - \mathcal{A}_x^{b\mu}\partial_\nu \mathcal{A}_x^{c\nu}\Big) - g^2 f^2_{ab,cd}\mathcal{A}_x^{b\nu}\mathcal{A}_x^{c\mu}\mathcal{A}_x^{d\nu} \\
& + \frac{i\hbar}{2} g f^{abc} \lim_{y\to x}\Big[\,(2\partial_\nu^y + \partial_\nu^x)(\psi_{x,y}^{c\nu,b\mu} + \psi_{y,x}^{b\mu,c\nu}) + \partial_\mu^x(\psi_{x,y}^{c\nu,b\nu} + \psi_{y,x}^{b\nu,c\nu})\Big] \\
& + i\hbar g^2 f^2_{ab,cd}\Big(\mathcal{A}_x^{b\nu}\psi_{x,x}^{\{d\nu,c\mu\}} + \mathcal{A}_x^{c\mu}\psi_{x,x}^{\{d\nu,b\nu\}} + \mathcal{A}_x^{d\nu}\psi_{x,x}^{\{b\nu,c\mu\}}\Big) \\
& + \hbar^2 g^2 f^2_{ab,cd}\psi'^{\,b\nu,c\mu,d\nu}_{x,x,x} + g f^{abc}\Big(\partial_\mu \bar{\mathcal{C}}_x^b c_x^c - i\hbar \lim_{y\to x}\partial_\mu^y \gamma_{x,y}^{c,b}\Big)\,,
\end{aligned}
$$

$$
\int_y K_{xy}^{ab}\mathcal{C}_y^b = \eta_x^a + g f^{abc}\Big(\partial_\mu \mathcal{A}_x^{b\mu}\mathcal{C}_x^c + \mathcal{A}_x^{b\mu}\partial_\mu \mathcal{C}_x^c - i\hbar\,\partial_\mu^x \theta_{x,x}^{c,b\mu}\Big)\,,
$$

$$
\int_y K_{xy}^{ab}\bar{\mathcal{C}}_y^b = \bar{\eta}_x^a + g f^{abc}\Big(\mathcal{A}_x^{b\mu}\partial_\mu \bar{\mathcal{C}}_x^c - i\hbar\,\partial_\mu^z \theta_{z,x}^{c,b\mu}\Big)\,.
$$

- It is possible to construct the quantum off-shell recursion relation by substituting the perturbiner expansion into above equations (one-loop level is done)

Conclusion

- We constructed the quantum perturbiner method for computing scattering amplitudes and correlation functions
- Quantum perturbiner expansion from the quantum effective action formalism by choosing the external field
- By substituting the perturbiner expansion into the DS equation, we have derived the quantum off-shell recursion relation for ϕ^4-theory and pure Yang–Mills theory
- It is a powerful technique that enables efficient computations of scattering amplitudes and correlation functions
- This formalism is applicable to any theory with an action, including gravity
- Interesting to compare with the generalised unitarity method

Thank you