A new mechanism for symmetry breaking from nilmanifolds

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Introduction and motivation

Exploring negative curvature compact geometries as a tool for model building at intermediate scale (relevant for unification and EW scales)

Important differences wrt flat compact extra dimensions
- Spectrum of the modes
- Symmetry breaking
- Scalar fields and potential, Yukawa terms generated by gauge interactions

Example: Gauge-Higgs unification models
- Standard approach uses tori as compact space (here negative curvature)
- Masses for the scalars are generated at loop level for the torus (here at tree level)

Nilmanifolds: general definition

The subalgebra $g_1 = [g_0, g_0]$ of g_0 consists of all linear combinations of Lie brackets of pairs of elements of g_0.

A Lie group G is solvable if its Lie algebra g terminates in the null algebra i.e. the sequence of its sub-algebras $g_0 = g$, $g_{n+1} = [g_n, g_n]$ for $n \geq 0$ reduces to the null algebra after a finite number of steps.

A Lie group G is nilpotent if the sequence $g_{n+1} = [g, g_n]$ reduces to the null algebra after a finite number of steps.

Therefore nilpotent groups are a special subclass of solvable groups. We refer to their corresponding compact manifolds as nilmanifolds.

Heisenberg Nilmanifold

nilpotent Lie algebra ex. Heisenberg algebra:

$$[Z_b, Z_c] = f^a{}_{bc} Z_a \longrightarrow [V_1, V_2] = -\mathtt{f} V_3 \ , \ [V_1, V_3] = [V_2, V_3] = 0$$

with f structure constant and coordinate system (Maurer-Cartan equation)

$$\mathrm{d}e^a = -\frac{1}{2} f^a{}_{bc}\, e^b \wedge e^c = -\sum_{b<c} f^a{}_{bc}\, e^b \wedge e^c$$

$$de^3 = \mathtt{f} e^1 \wedge e^2 \ ; \ de^1 = 0 \ ; \ de^2 = 0$$

Heisenberg Nilmanifold

$$e^1 = r^1 dx^1 \;;\quad e^2 = r^2 dx^2 \;;\; e^3 = r^3 \left(dx^3 + N x^1 dx^2\right)$$

$$\text{where} \quad N = \frac{r^1 r^2}{r^3} \mathtt{f} \in \mathbb{N} \,.$$

The general metric can be obtained from the interval:

$$\mathrm{d}s^2 = \left(e^1 + a e^3\right)^2 + \left(e^2 + b e^3\right)^2 + c^2 \left(e^3\right)^2 \,, \quad a, b \in \mathbb{R}, \;\; c \in \mathbb{R}^*$$

and volume is

$$V = \int \mathrm{d}^3 x \sqrt{g} = r^1 r^2 r^3 |c|$$

the Ricci scalar is always negative

$$\mathcal{R} = -\frac{1}{4} \delta_{ad} \delta^{bc} \delta^{eg} f^a_{be} f^d_{cg} \quad \longrightarrow \quad \mathcal{R} = -\frac{1}{2} \mathtt{f}^2$$

Why Heisenberg nilmanifold?

Heisenberg manifold ⇔ 2-torus with twisted circle fiber
Calculable spectrum of the Laplace operator $\Delta f = \lambda f$
Eigenfunctions form a complete set on the space:

$$f(x) = \sum_i c_i U_i(x)$$

$$\Delta B_m = \lambda B_m$$

Eigenscalars (U_i) and one-forms (B_m) have analytical expressions
To make the manifold compact:

$$x^1 \sim x^1 + n^1 \; ; \; x^2 \sim x^2 + n^2 \; ; \; x^3 \sim x^3 + n^3 - n^1 N x^2$$
$$n^1, n^2, n^3 \in \{0, 1\} \; .$$

From 7D Yang-Mills to 4D

The effective action is computed from the 7D YM action :

$$\mathcal{L}_{4D} = \int dy^3 \mathcal{L}_{7D} \; ; \; \mathcal{L}_{7D} = \frac{1}{2} \mathrm{Tr}\left(F_{MN} F^{MN}\right)$$

$$\begin{aligned} \mathcal{A}^a &= \mathcal{A}^a_M(x^M)\mathrm{d}x^M = \mathcal{A}^a_\mu(x^M)\mathrm{d}x^\mu + \mathcal{A}^a_m(x^M)\mathrm{d}x^m \\ &= \sum_I U_I(x^m) A^{aI}_\mu(x^\mu)\mathrm{d}x^\mu + \phi^{aI}(x^\mu) B_{Im}(x^m)\mathrm{d}x^m \end{aligned}$$

where U_I and B_I are respectively 3d eigenscalars and 3d eigen-one-forms of the Laplacian on the nilmanifold, while A^{aI} and ϕ^{aI} are a 4d one-form and a 4d scalar.

From 7D Yang-Mills to 4D

The sum over I is an infinite multi-index sum over the basis of 3d eigen-forms, the geometrical limit ("large base, small fiber" limit)

$$|\mathrm{f}| \ll \frac{1}{r^i}\ , \ i = 1, 2, 3 \quad \Rightarrow \quad r^3 \ll r^{1,2}$$

separates the low-lying masses from the rest of the tower

The resulting 4D action "generates" a scalar part:

$$S = \int \mathrm{d}x^4 \mathrm{Tr}\Big(- \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{3} D_\mu \phi^i D^\mu \phi^i - M^2 (\phi^3)^2 - \mathcal{U} \Big)$$

Lowest modes

Scalars:

$$U_{I=1} = \frac{1}{\sqrt{V}} \; ; \quad \lambda_{U_1} = 0$$

one-forms:

$$B_{I=1} = \frac{1}{\sqrt{V}} e^1 \; ; \quad \lambda_{B_1} = 0$$

$$B_{I=2} = \frac{1}{\sqrt{V}} e^2 \; ; \quad \lambda_{B_2} = 0$$

$$B_{I=3} = \frac{1}{\sqrt{V}} e^3 \; ; \quad \lambda_{B_3} = \mathtt{f}^2$$

other modes:

$$(m_{tower})^2 \sim \frac{1}{(r^i)^2}$$

Scalar potential

$$\mathcal{U} = \mathsf{Tr}\Big(- 2ig M[\phi^1, \phi^2]\phi^3 + \tfrac{1}{2}g^2 \sum_{i,j=1}^{3} [\phi^i, \phi^j][\phi^i, \phi^j] \Big)$$

with M = |f| and g = g7/$\sqrt{V}$; scalars in the adjoint representation. We have to minimise the potential (mass + interaction part)

$$\frac{\mathcal{V}}{M^2} = \mathrm{Tr}(\phi^3)^2 + \frac{\mathcal{U}}{M^2}$$

computing $\delta\mathscr{V}/M^2$ and the masses from $\delta^2\mathscr{V}/M^2$

Vacuum condition : $\phi_3 = 0$; $[\phi_1, \phi_2] = 0$

$\Rightarrow$ Pick $\phi_1, \phi_2 \in$ Cartan sub-algebra.

The mass matrix is block diagonal in root space

Once the mass matrix is diagonalized, the masses for a given root Eα are :

$$0\ ,\ (m_\alpha^{\pm})^2 = \frac{1}{2} M^2 \left(1 + 2((b_1^\alpha)^2 + (b_2^\alpha)^2)) \pm \sqrt{1 + 4((b_1^\alpha)^2 + (b_2^\alpha)^2))}\right)$$

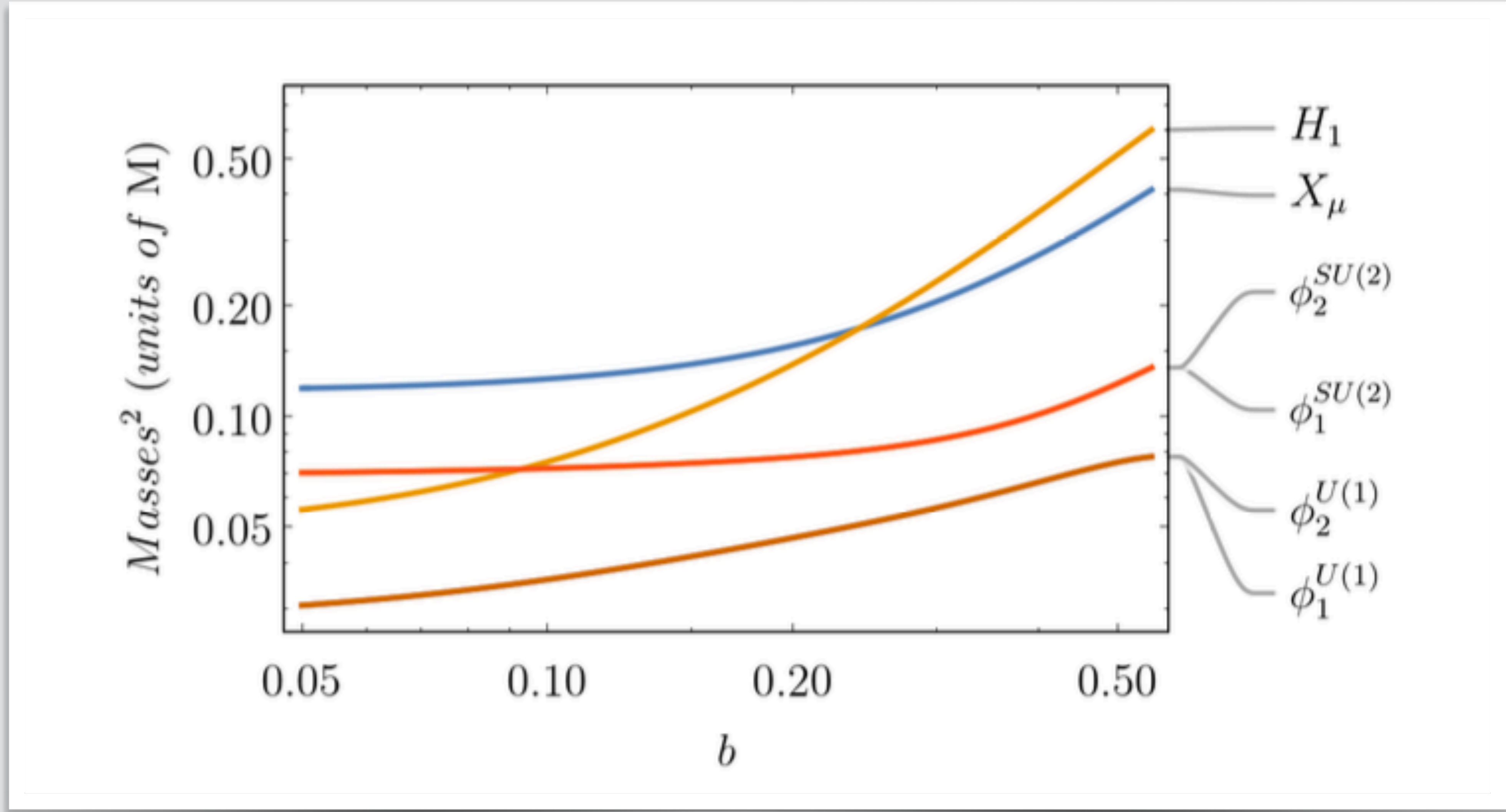
lifted by loop corrections

gauge boson masses

$$m^2_{\alpha,gauge} = \mathrm{g}^2 \sum_{I=1}^{2} \phi_0^{Ii} \alpha_i = M^2 \left((b_1^\alpha)^2 + (b_2^\alpha)^2\right)$$

In this convention the vacuum parameters b_i are dimensionless, so that b_i = Mg $\tilde{b}_i$, and $\tilde{b}_i$ has mass dimension one. In the following $b_1=b_2=b$

SU(3) -> SU(2)xU(1)



One-loop renormalized masses of the low-mass scalars for the SU(3) breaking pattern. H_1 and X_μ are in the fundamental representation of SU(2) × U(1), while $\phi_{SU(2)}$ is the adjoint of SU(2) and $\phi_{U(1)}$ in the adjoint of U(1).

Dark matter and orbifolding

The scalar spectrum for the KK-type modes has the form:

$$M^2_{k,l,n} = (2\pi k)^2 \left(1 + \frac{2n+1}{2\pi|k|}\right)$$

Using the discrete symmetries T and P:

$$T: \quad x^{1,2} \to -x^{1,2} \; ; \quad x^3 \to x^3$$

$$P: \quad x^1 \leftrightarrow x^2 \; ; \quad x^3 \to -x^3 - N x^1 x^2$$

we have $M_3/(Z_2 x Z_2)$ and localising the SM particles at the origin is consisten
with the orbifolding.

Dark matter and orbifolding

The orbifold and DM symmetries are:

$$\text{orbifold:} \quad \rightarrow x^1 \leftrightarrow x^2 \,, \quad x^3 \rightarrow -x^3 - N x^1 x^2$$
$$\text{DM parity:} \rightarrow x^{1,2} \leftrightarrow -x^{1,2} \,, \quad x^3 \rightarrow x^3 \,.$$

the modes can be classified accordingly:

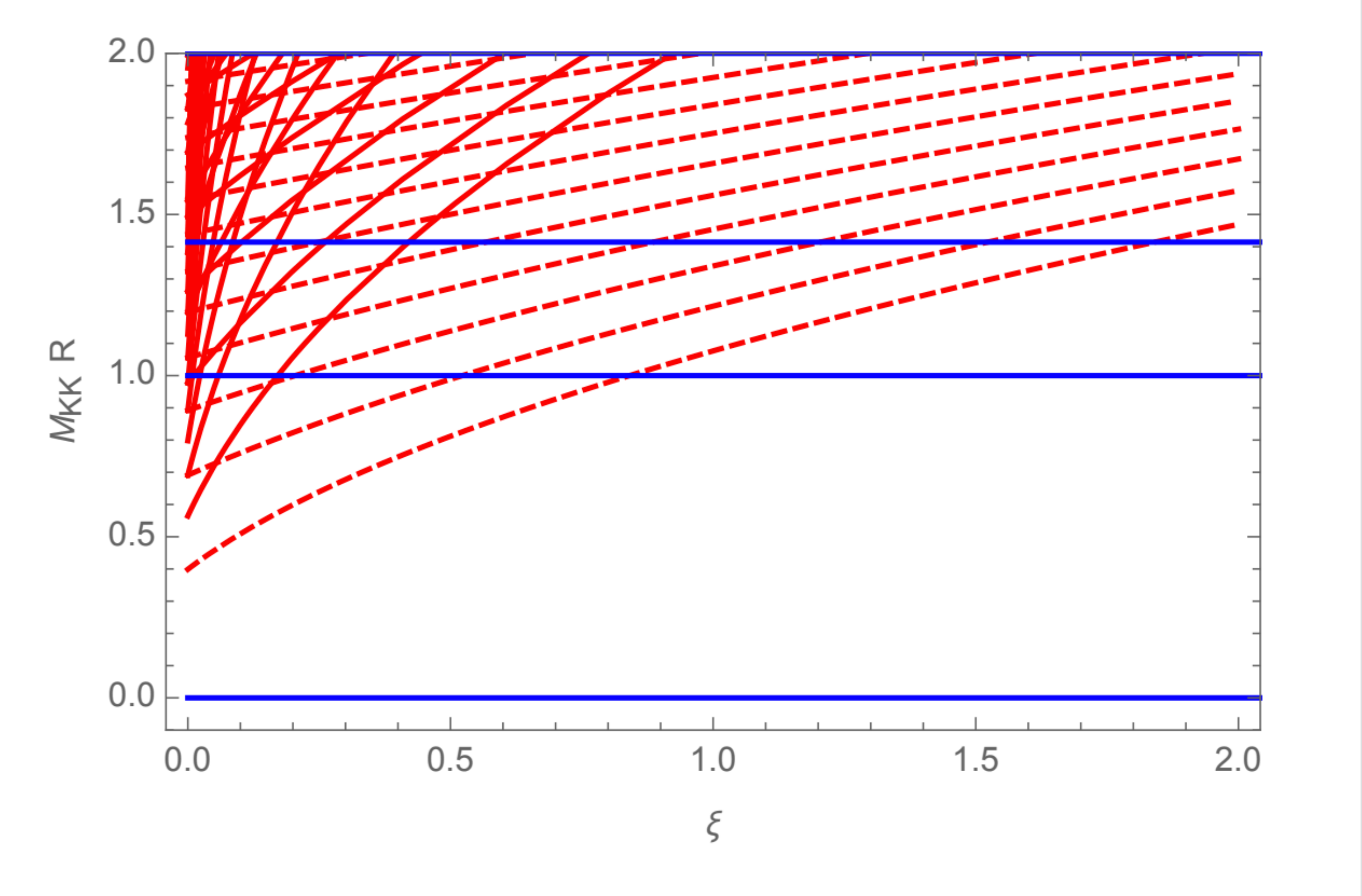
	$M_{KK}^2 \left(\frac{r}{2\pi}\right)^2$	orbifold even DM-even	orbifold even DM-odd	orbifold odd DM-even	orbifold odd DM-odd
(l,n)	Torus modes				
$(0,0)$	0	1	-	-	-
$(l,0)$	l^2	1	1	1	1
(l,l) & $(l,-l)$	$2l^2$	1	1	1	1
$(l,\|n\|)$ & $(l,-\|n\|)$	l^2+n^2	2	2	2	2
(k,n)	Fiber modes				
even k	$\|N\|\left(\frac{k(2n+1)}{2\pi}+k^2\xi\right)$	$k/2$	$k/2$	$k/2$	$k/2$
odd k		$(k+1)/2$	$(k-1)/2$	$(k+1)/2$	$(k-1)/2$

Dark matter candidate

The lightest parity odd particle is stable and is the DM candidate. If:

$$\xi = \frac{1}{|N|}\left(\frac{r}{cr^3}\right)^2 \qquad \xi < \frac{\pi - |N|}{4\pi|N|} \; (= 0.17 \;\; \text{for } N = 1)$$

the fiber mode is the lightest, otherwise the torus mode.



in red the fiber modes (dashed have no odd part) and in blue the torus ones

Fermions

Similar to the Laplacian it is possible to compute the spectrum of the Dirac operator (here simple case a=b=0 shown) :

$$(D-\lambda)\psi = 0 \; ; \quad D \equiv \gamma^a e^m{}_a(\partial_m + \tfrac{1}{4}\omega_{mbc}\gamma^{bc}),$$

$$D = \gamma^a V_a + i\frac{\mathtt{f}}{4},$$

where ω_{mbc} is the spin connection and for M3 ω_{123} = f/2. Solutions are:

$$-\mathrm{i}\lambda_{k,l,n} = \frac{f}{4} \pm \sqrt{\left(\frac{\sigma}{f}\right)^2 + |\sigma|(p^+(\sigma)2n + p^-(\sigma)2(n+1))}.$$

$$\psi_{k,l,n} = C\begin{pmatrix} u_{k,l,n} \\ p^+(\sigma)\alpha u_{k,l,n-1} + p^-(\sigma)\beta u_{k,l,n+1}\end{pmatrix},$$

σ and $p^{+/-}(\sigma)$ contain the parameters a, b and f

Reduction to 4 dimensions

Similar to the pure Yang-Mills case we can obtain the 4D effective theory from the 7D one:

$$\mathcal{L}_{4D}^{\text{eff}} = \int \mathrm{d}^3 y \, \left(\mathcal{L}_{\text{7D}}^{\text{YM}} + \mathcal{L}_{\text{7D}}^{\text{f}}\right)$$

possible flux term

with

$$\mathcal{L}_{\text{7D}}^{\text{f}} = \overline{\psi}_i \Gamma^M (\delta^{ij} \nabla_M + i \mathcal{A}_M^a \rho_a^{ij}) \psi_j + \tfrac{1}{6} F_{MNP} \overline{\psi}_i \Gamma^{MNP} \psi_i + M_0 \overline{\psi}_i \psi_i,$$

representation R of the gauge group

also in the representation R
4d Weil spinor non-chiral

Ansatz for the 7D spinors:

$$\psi_i = (\chi_{i+} + \theta_{i-}) \otimes \xi,$$

Reduction to 4 dimensions

We have:

$$\int \mathrm{d}^3 y \; \mathcal{L}^{\mathrm{f}}_{7\mathrm{D}} = \mathcal{L}_{\mathrm{kinetic}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{mass}},$$

with

$$\begin{aligned}
\mathcal{L}_{\mathrm{kinetic}} &= \overline{\chi}_{i+}\gamma^{\mu}(\delta^{ij}\partial_{\mu} + igA^{a}_{\mu}\rho^{ij}_{a})\chi_{j+} + \overline{\theta}_{i-}\gamma^{\mu}(\delta^{ij}\partial_{\mu} + igA^{a}_{\mu}\rho^{ij}_{a})\theta_{j-} \\
\mathcal{L}_{\mathrm{Yukawa}} &= ig(\overline{\theta}_{i-}\chi_{j+} - \overline{\chi}_{i+}\theta_{j-})\rho^{ij}_{a}\Phi^{a} \\
\mathcal{L}_{\mathrm{mass}} &= M_c\, \overline{\theta}_{i-}\chi_{i+} + M^{*}_{c}\, \overline{\chi}_{i+}\theta_{i-},
\end{aligned}$$

the Yukawa Lagrangian is generated from the gauge interactions!
As usual here the “gauge” scalar is in the adjoint representation

Conclusions

- Twist f ⇔ Mass at tree level M
- Potential allows for various symmetry breaking
- Model is rigid (Yang-Mills in 7D + nilmanifold)
- Analytical results all the way for any gauge group G (but G may be constrained by physical motivations)
- Moduli of the metric on the Heisenberg manifold computed
- Laplacian spectrum for scalars/vectors with arbitrary metric solved for the lowest modes
- Dirac operator with arbitrary metric solved, fermions can be considered both in 7D (bulk) and 4D (localised)
- Exiting playground for model building!