UNIVERSITY OF LJUBLJANA
FMF
Faculty of Mathematics and Physics

IJS

False Vacuua From Thin to Thick Walls

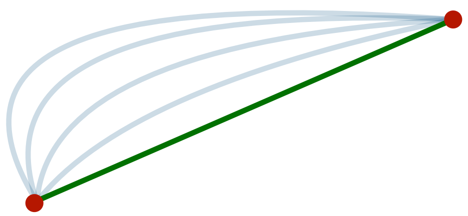
Joint Yonsei-Konkuk-Sogang Mini-workshop on FOPT

Miha Nemevšek, 6 February 2025

aris
Slovenian Research and Innovation Agency

Outline

Introduction, phase transitions, formalism

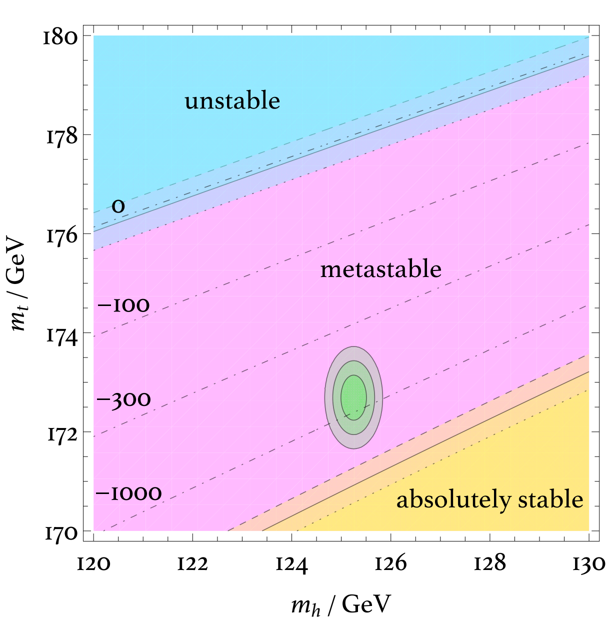


Bounces, thin and thick walls

Guada, Maiezza, MN '18
Guada, MN, Pintar '20

Fluctuations, functional determinants

Ivanov, Matteini, MN, Ubaldi '22
Matteini, MN, Shoji, Ubaldi '24

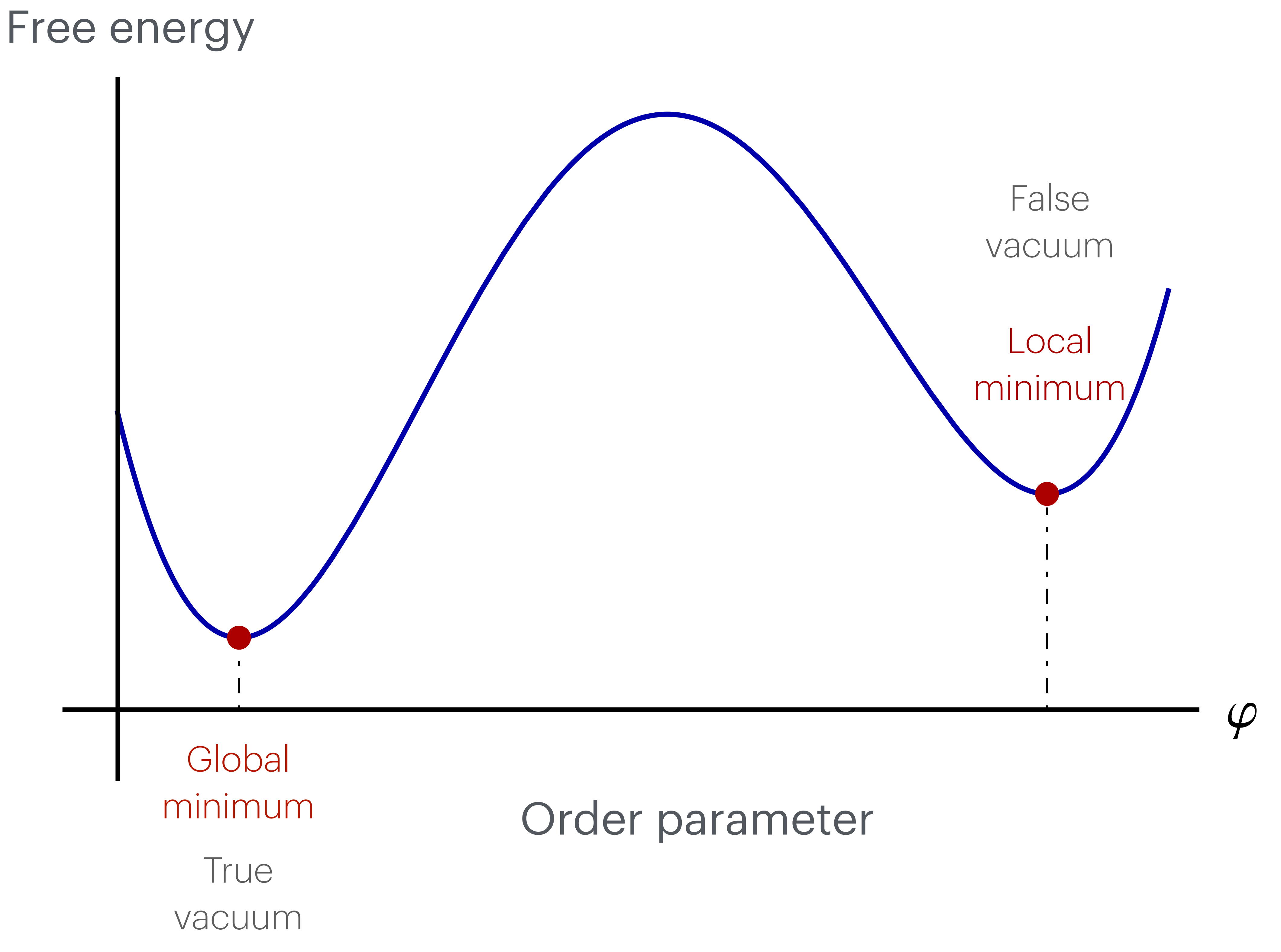


* Standard Model Decay Rate

Baratella, MN, Shoji, Trailović, Ubaldi '24

* not this talk, but happy to discuss

Phase transitions



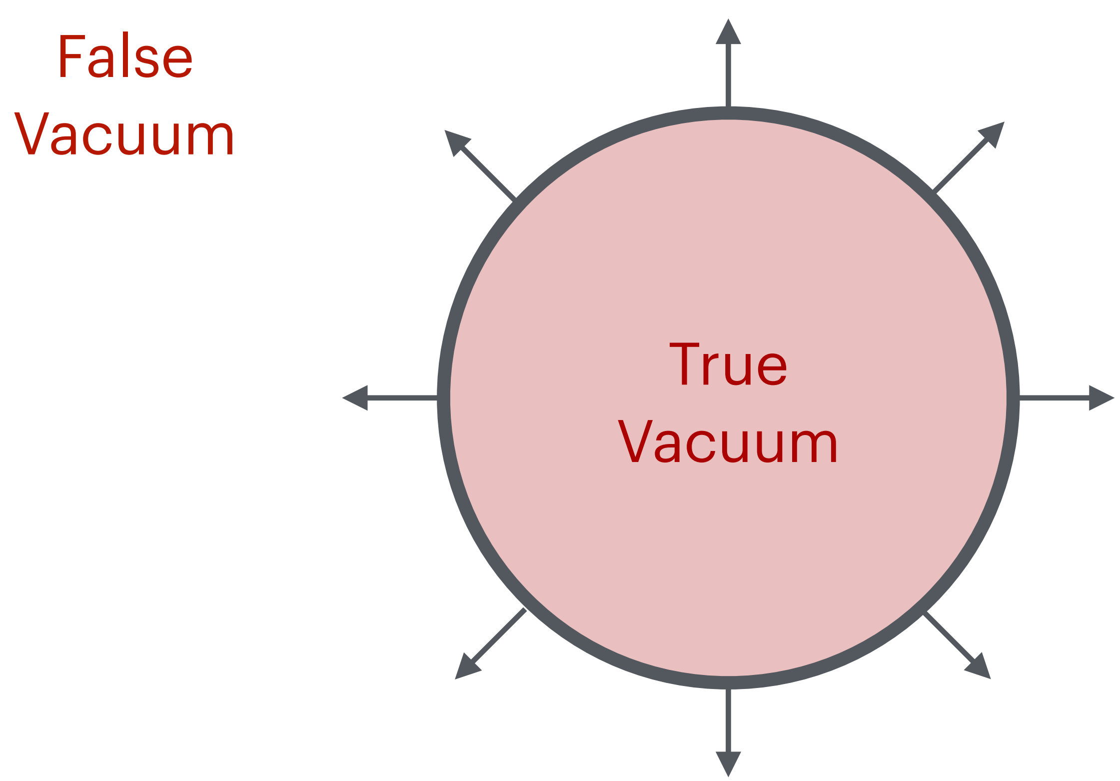
Bubble nucleation

Barrier implies a first order phase transition

QM

$$\Gamma = A\, e^{-B}$$

Gamow '28



α decay

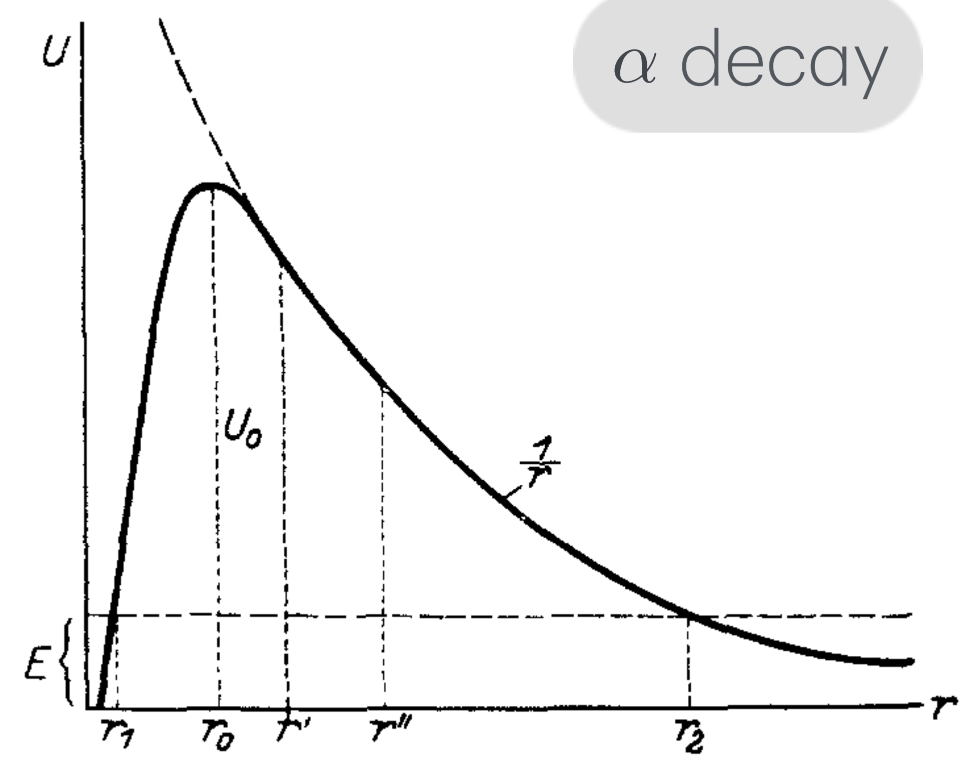


Fig. 1.

Bubble nucleation

Barrier implies a first order phase transition

exponential

$$\Gamma = A\, e^{-B}$$

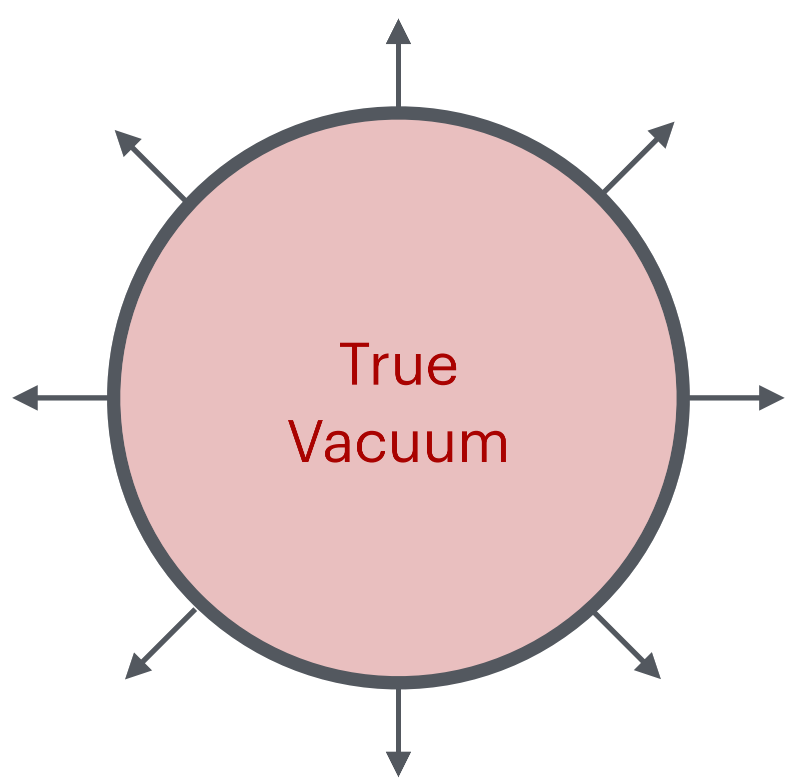
QM

Gamow '28

WKB

α decay

False Vacuum



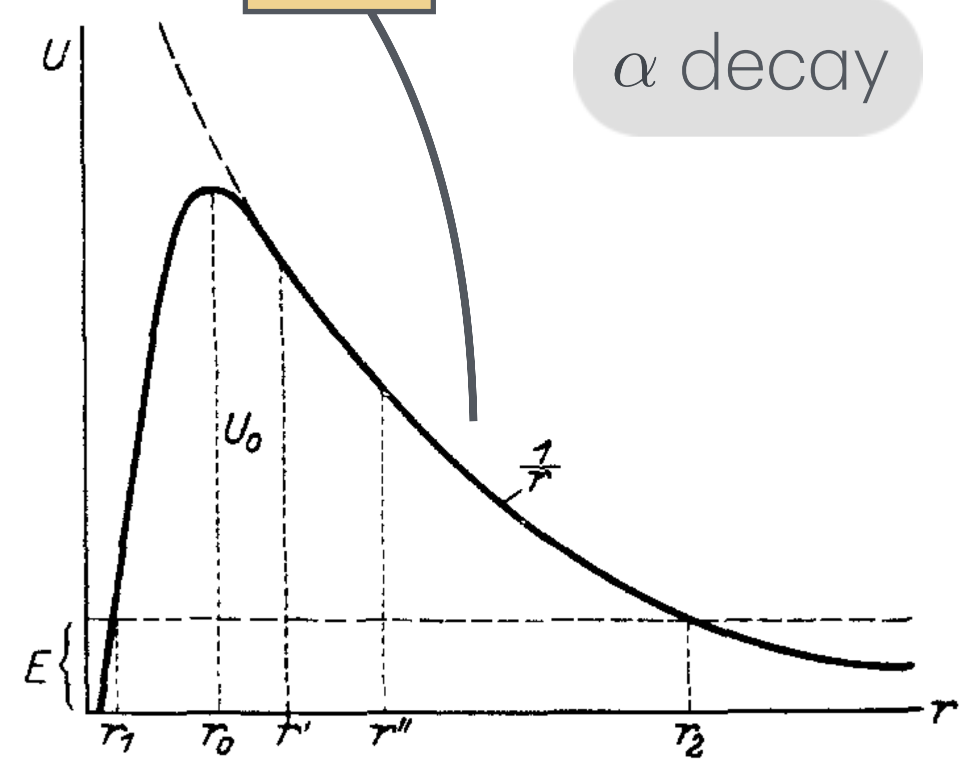


Fig. 1.

Bubble nucleation

Barrier implies a first order phase transition

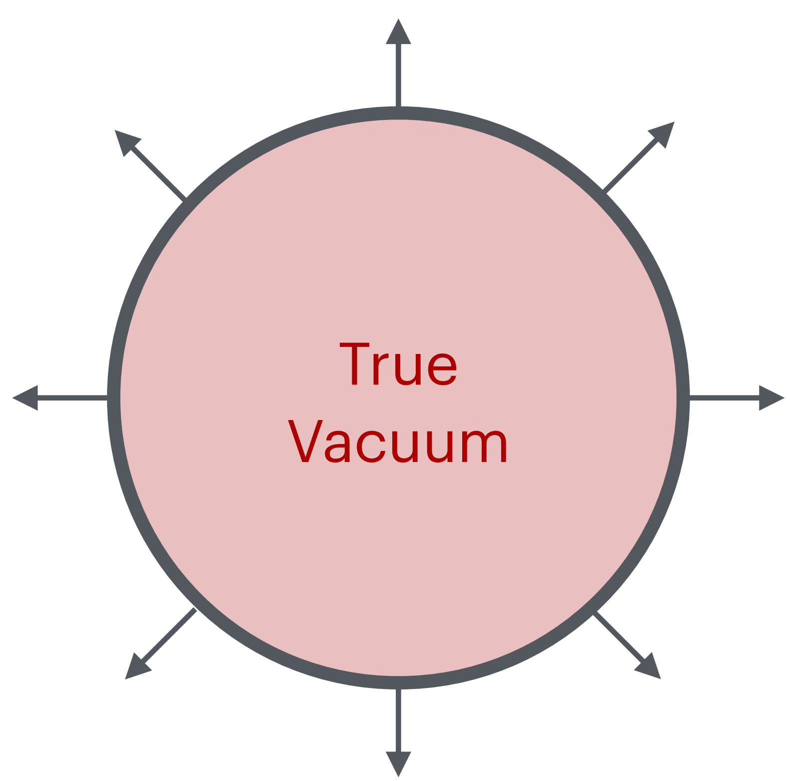
prefactor

$$\Gamma = A e^{-B}$$

QM

Gamow '28

False Vacuum



$\sim E_n$

α decay

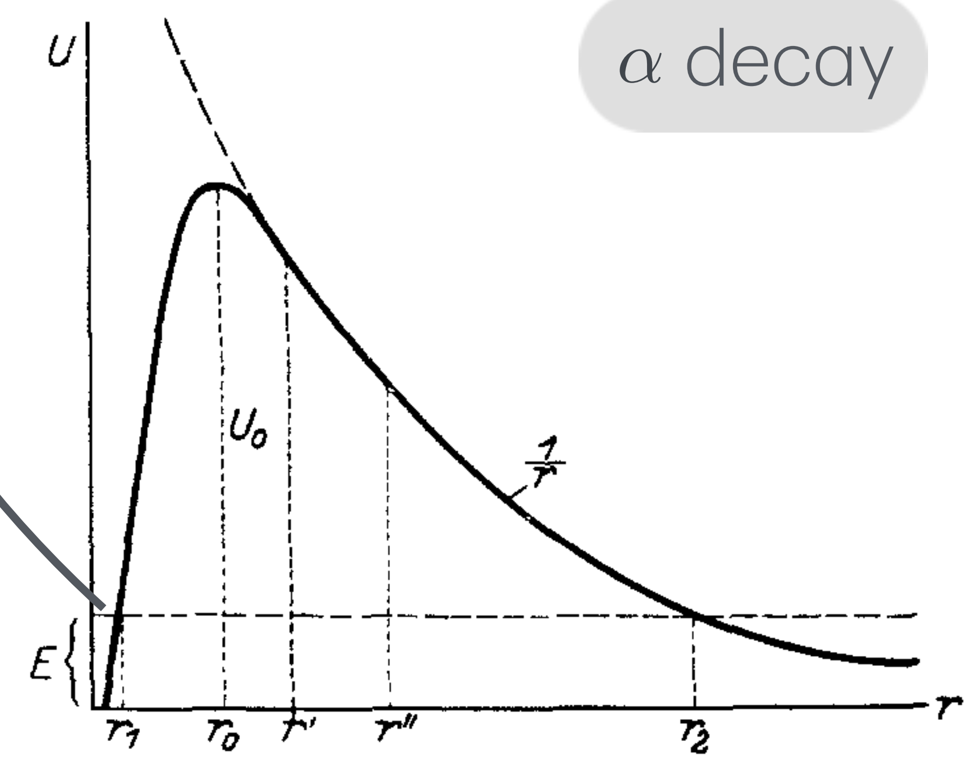


Fig. 1.

Motivation

QFT rates in physical systems, flat spacetime

$D = 4$ Zero temperature QFT, vacuum stability

SM lifetime, BSM constraints on parameters λ_i, y_i

α_s, m_t, m_h

$D = 3$ Finite temperature TFT, existence of new phases

Cosmology: GWs, genesis, primordial B-fields, inflation...

$D = 2$ Topological defects, condensed matter, more?

curved ST + gravity...

Motivation

QFT rates in physical systems, flat spacetime

$D = 4$ Zero temperature QFT, vacuum stability

SM lifetime, BSM constraints on potential parameters

This talk (mostly)

$D = 3$ Finite temperature TFT, existence of new phases

GWs, genesis, primordial B-fields, inflation...

$D = 2$ Condensed matter, spin chains, more?

curved ST + gravity...

Formalism

Minkowski to Euclidean $t^2 - x_i^2 \to t^2 + x_i^2 = \rho^2$

+ alternative derivations, not this talk

..., Devoto, Devoto, Di Luzio, Ridolfi '22

$$\frac{\Gamma}{V} = \frac{\mathrm{Im} \int \mathcal{D}\phi \, e^{-S[\phi]}}{\int \mathcal{D}\phi \, e^{-S[\hat{\phi}]}}$$

Coleman '77
Callan, Coleman '77

Euclidean action $S[\phi] = \int \mathrm{d}^D x \left(\frac{1}{2} \partial\phi^2 + V(\phi) \right)$

Saddle point field configuration: bounce $\overline{\phi}$

Formalism

Minkowski to Euclidean $t^2 - x_i^2 \to t^2 + x_i^2 = \rho^2$

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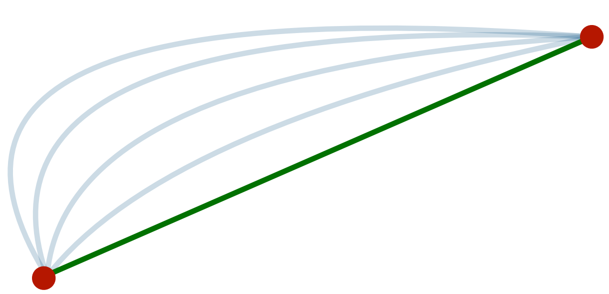
$$S[\phi] \simeq S[\overline{\phi}] + \left.\frac{\delta S}{\delta \phi}\right|_{\overline{\phi}} \delta\phi + \frac{1}{2} \left.\frac{\delta^2 S}{\delta \phi^2}\right|_{\overline{\phi}} \delta\phi^2 + \dots$$

Bounce action



$= 0$

Formalism

Minkowski to Euclidean $t^2 - x_i^2 \to t^2 + x_i^2 = \rho^2$

+ alternative derivations, not this talk

..., Devoto, Devoto, Di Luzio, Ridolfi '22

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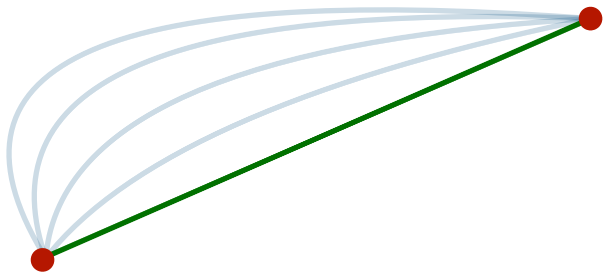
$$S[\phi] \simeq S[\overline{\phi}] + \underbrace{\frac{\delta S}{\delta \phi}\bigg|_{\overline{\phi}} \delta\phi}_{= 0} + \boxed{\frac{1}{2} \frac{\delta^2 S}{\delta \phi^2}\bigg|_{\overline{\phi}} \delta\phi^2} + \ldots$$

+ fluctuations

Semi-classics



$$\delta S = 0 \to \overline{\varphi}(\rho)$$

Polygonal bounces
FindBounce

2nd talk

Guada, Maiezza, MN '18
Guada, MN, Pintar '20

Thin wall

Ivanov, Matteini, MN, Ubaldi '22

Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Bounce

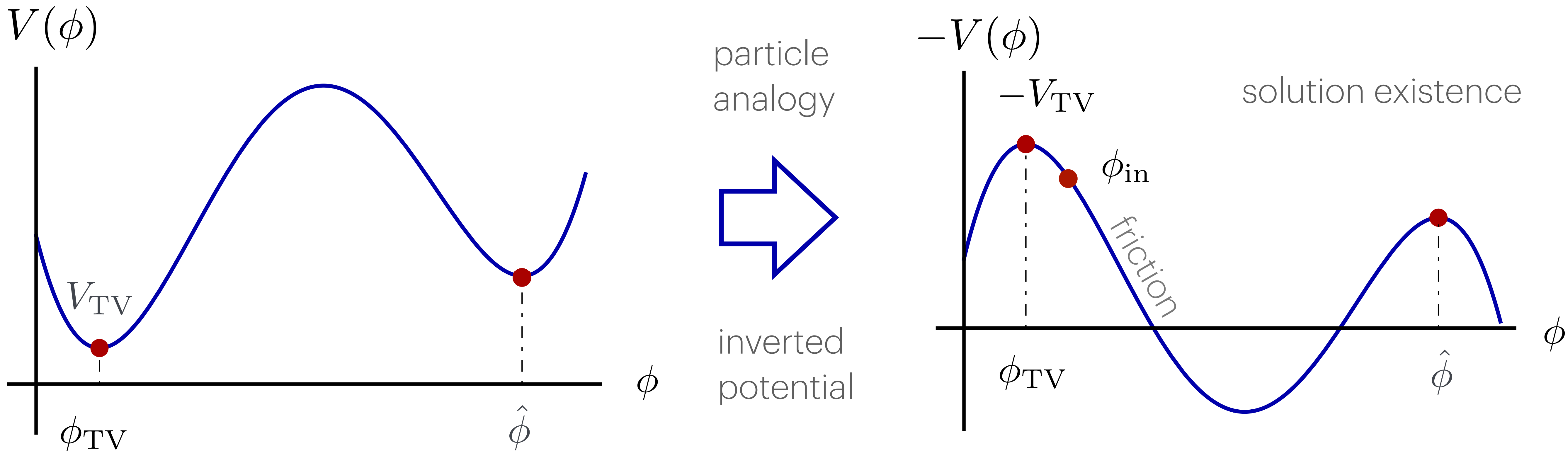
Coleman '77

Action $$S = \Omega \int_0^\infty \mathrm{d}\rho\, \rho^{D-1} \left(\frac{1}{2}\dot{\phi}^2 + V - \hat{V} \right) , \qquad \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Extremize $$\ddot{\phi} + \frac{D-1}{\rho}\dot{\phi} = \frac{\mathrm{d}V}{\mathrm{d}\phi}$$

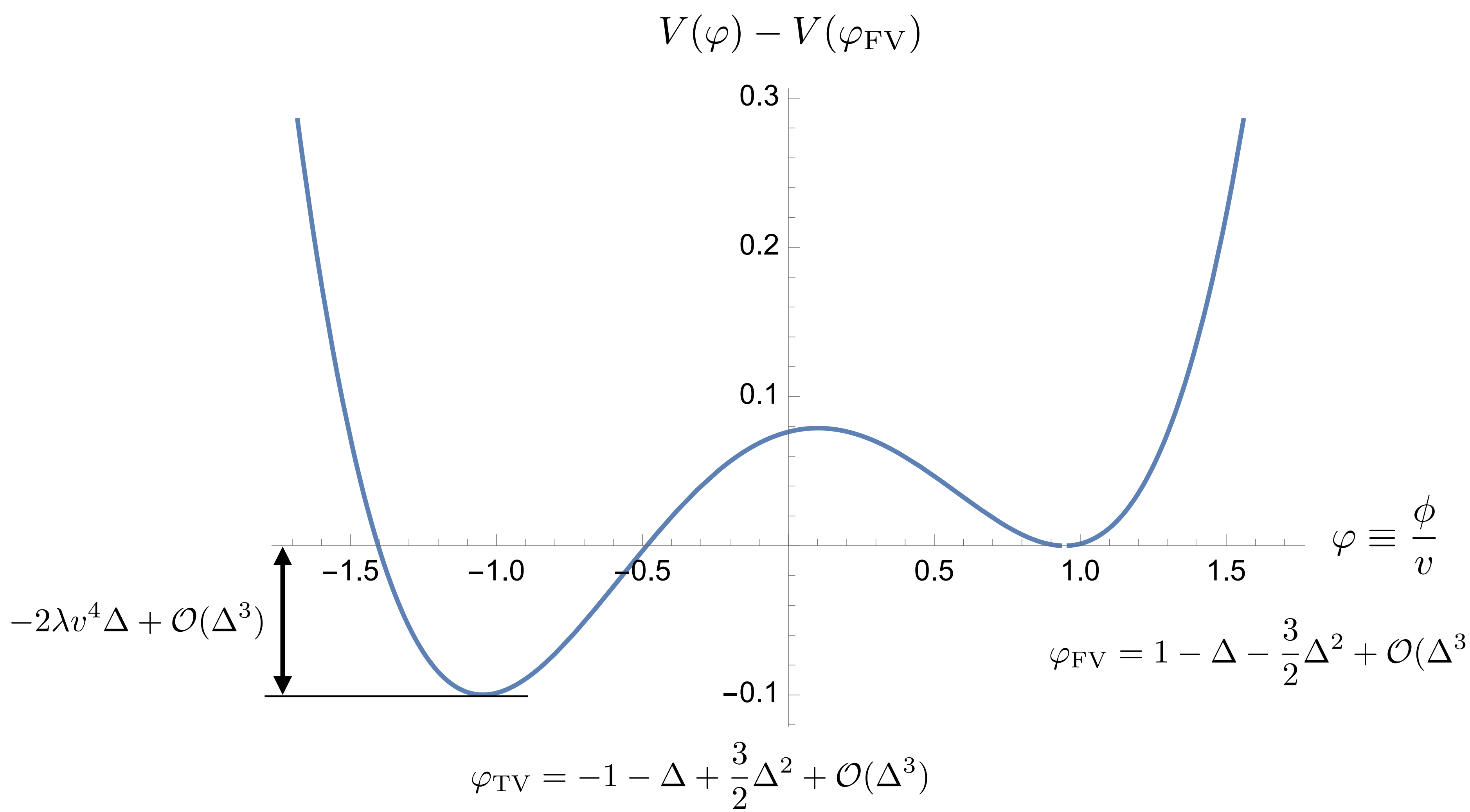
$$\phi(0) = \phi_{\text{in}} , \quad \phi(\infty) = \hat{\phi}$$

$$\dot{\phi}(0) = \dot{\phi}(\infty) = 0$$



$$V = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 (\phi - v)$$

Overall coupling	Scale of the model	TW parameter
$[\lambda] = 4 - D$	$[v] = D/2 - 1$	$[\Delta] = 0$



Scale of the model $\quad 0 < \lambda \ll 1, \quad 0 < \Delta \ll 1, \quad \Delta_{\max} = 3^{-3/2} \quad$ TW = near degenerate

$$V = \frac{\lambda}{8}\left(\phi^2 - v^2\right)^2 + \lambda \Delta v^3 (\phi - v)$$

Ivanov, Matteini, MN, Ubaldi '22

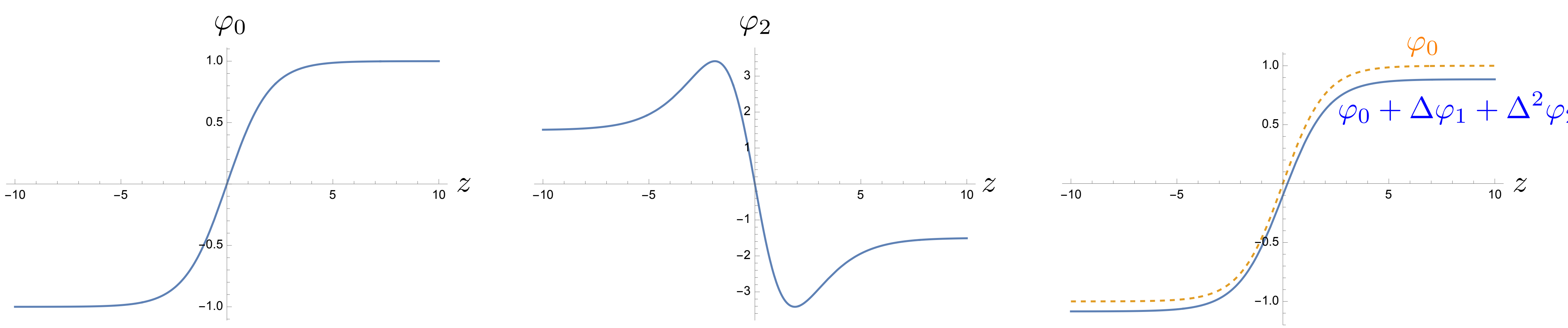
Expand around the bounce radius

$$\varphi(z) = \sum \varphi_n(z) \Delta^n\,, \qquad z = \sqrt{\lambda} v\, \rho - r\,, \qquad r = \frac{1}{\Delta} \sum r_n \Delta^n$$

Solve the bounce equation

$$\varphi_0 = \mathrm{th}\frac{z}{2}\,, \qquad \varphi_1 = -1\,,$$

$$r_0 = \frac{D-1}{3}\,, \qquad r_1 = 0$$



See the paper for φ_2, φ_3, physical bubble profile

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}$$

Thin Wall Action

Ivanov, Matteini, MN, Ubaldi '22

λ and v factor out

$$S = \frac{\Omega\, v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \times \tilde{S}[\Delta^2]$$

Leading order

$$S_0 = \frac{\Omega\, v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \left(\frac{D-1}{3}\right)^{D-1} \frac{2}{3D}$$

2nd order

$$S_2 = S_0 \left(1 + \Delta^2 \left(\frac{1 + D\,(25 - 8D - 3\pi^2)}{2(D-1)}\right) + \mathcal{O}(\Delta^4)\right)$$

$D = 4$: FV decay at T = 0 — Coleman '77

$D = 3$: FV nucleation at finite T — Affleck '81, Linde '83

$$S_2 = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1\right)\Delta^2\right), & D = 3\,, \\ \frac{\pi^2}{3\lambda}\left(1 - \left(2\pi^2 + \frac{9}{2}\right)\Delta^2\right), & D = 4 \end{cases}$$

From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

$$S = \Omega \frac{v^{4-D}}{\lambda^{D/2-1}} S_L(\Delta)$$

Study the series $$S_L^{(N)}(\Delta) = S_L^{(0)} \left(1 + \sum_{n=1}^{N/2} \Delta^{2n} s_{2n}^L \right)$$

Analytics all the way to Δ^4

$$s_2^L = \frac{-8D^2 + (25 - 3\pi^2)D + 1}{2(D-1)}$$

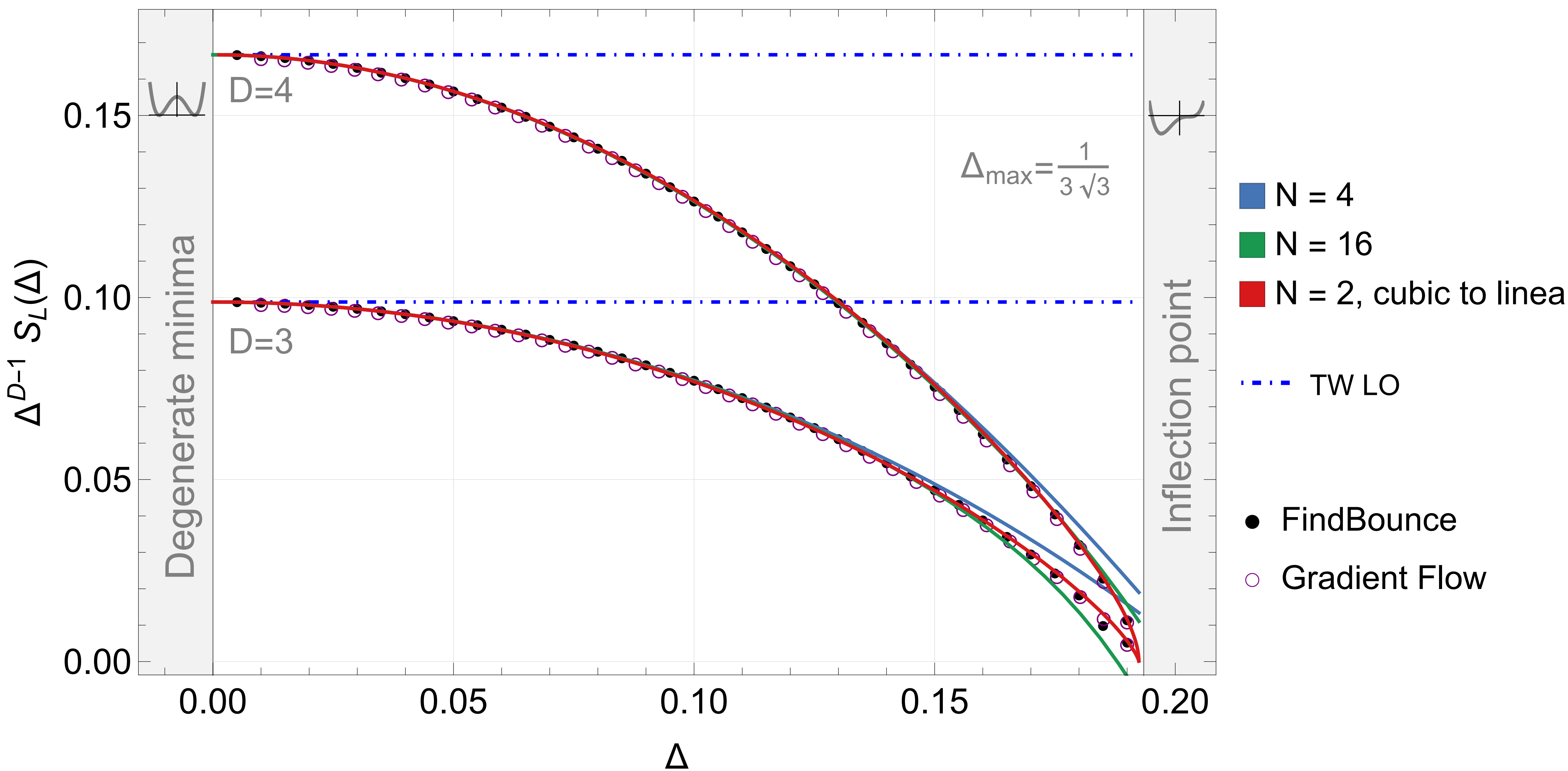
$$s_4^L = \frac{1}{40(D-1)^3} \Big(320D^5 + 80D^4\left(3\pi^2 - 49\right) - 3D^3\left(550\pi^2 + 3\pi^4 - 6185\right) + 5D^2\left(426\pi^2 + 45\pi^4 - 648\zeta(3) - 7843\right) + D\left(3240\zeta(3) + 30635 + 360\pi^2 - 414\pi^4\right) + 105 \Big)$$

From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Semi-analytic: fix D numerically, go up to Δ^{40}

	s_2^L	s_4^L	s_6^L	s_8^L	s_{10}^L	s_{12}^L	s_{14}^L	s_{16}^L
$D=3$	-21.2	-57.6	-977	$-2.01\cdot10^4$	$-4.73\cdot10^5$	$-1.24\cdot10^7$	$-3.65\cdot10^8$	$-1.23\cdot10^{10}$
$D=4$	-24.2	7.53	-266	$-5.86\cdot10^3$	$-1.21\cdot10^5$	$-2.50\cdot10^6$	$-5.08\cdot10^7$	$-9.34\cdot10^8$



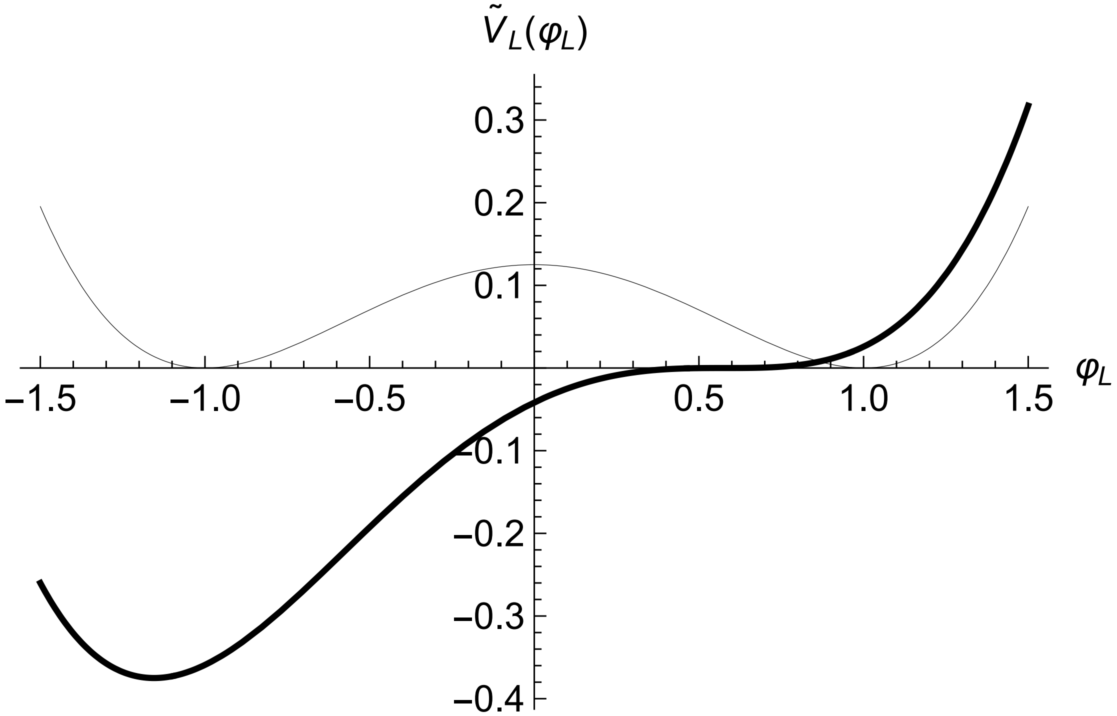
From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Parametrization dependence of TW expansion

Linear

$$V_L(\phi_L) = \frac{\lambda}{8}\left(\phi_L^2 - v^2\right)^2 + \lambda \Delta v^3 \left(\phi_L - v\right)$$

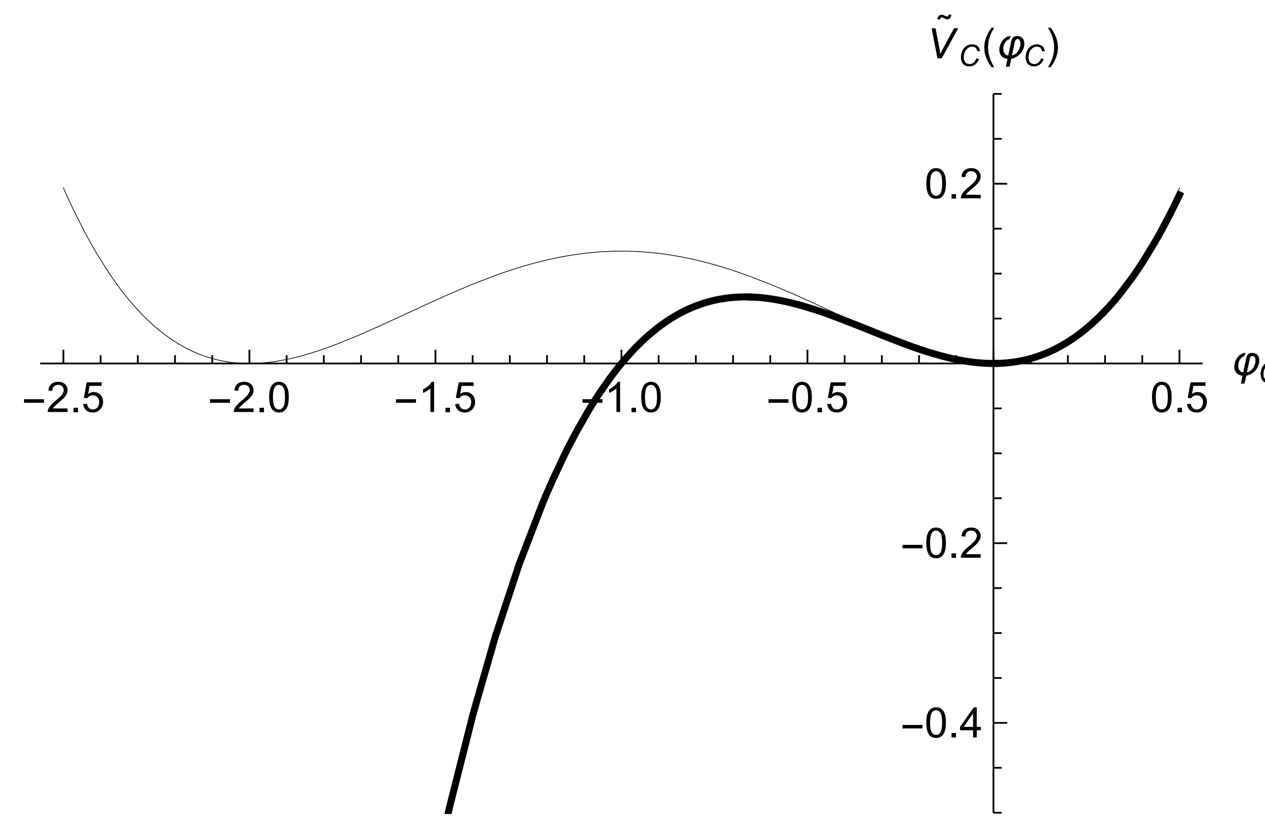


$$S = \Omega \frac{v^{4-D}}{\lambda^{D/2-1}} S_L(\Delta)$$

Cubic

$$\varepsilon_\alpha \equiv 1 - \lambda_C \frac{m^2}{4\eta^2}$$

$$V_C(\phi_C) = \frac{1}{2} m^2 \phi_C^2 + \eta \phi_C^3 + \frac{1}{8}\lambda_C \phi_C^4$$



$$S = \Omega \frac{m^{4-D}}{\lambda_C}(1 - \varepsilon_\alpha) S_C(\varepsilon_\alpha)$$

From Linear to Cubic (and back again)

Matteini, MN, Shoji, Ubaldi '24

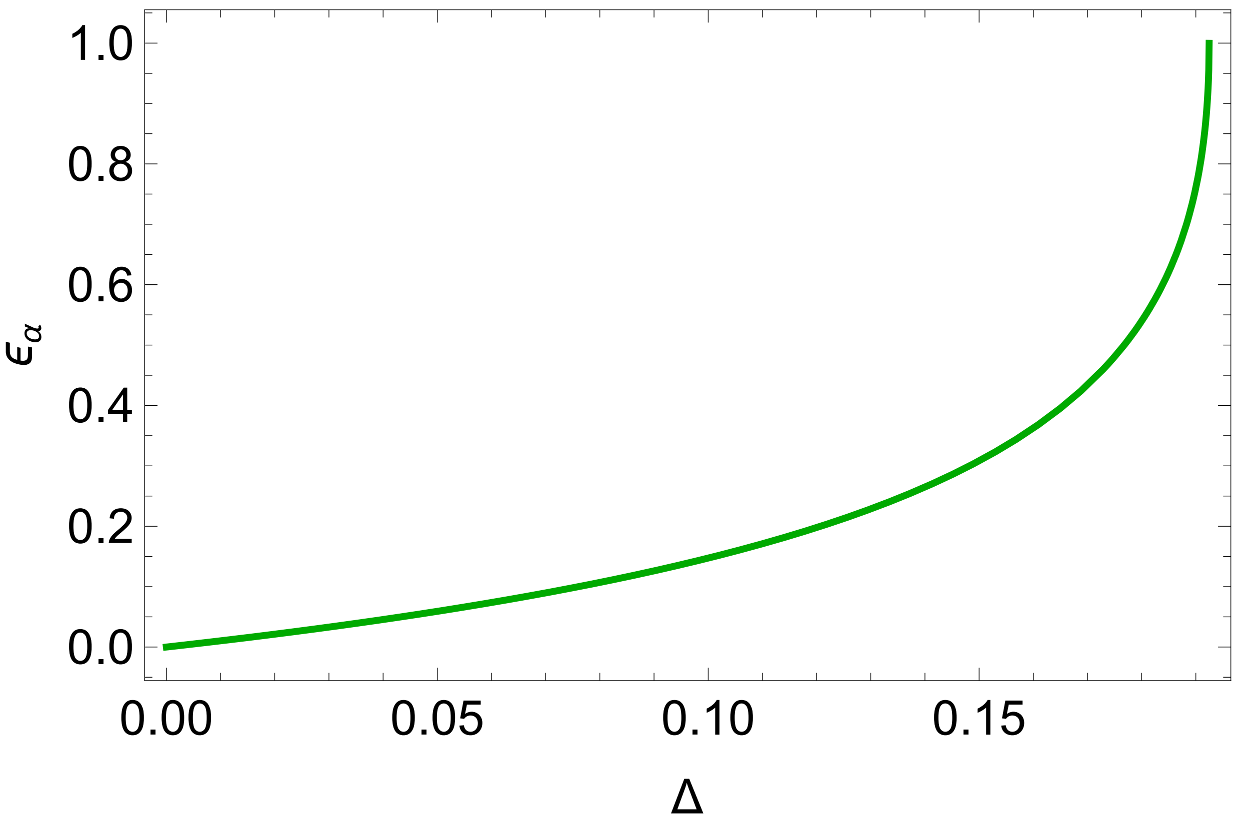
$$L \to C$$

$$\{\lambda, v, \Delta\} \to \{\epsilon_\alpha, m, \eta\}$$

$$V_L(\phi_L) = \frac{\lambda}{8}\left(\phi_L^2 - v^2\right)^2 + \lambda\Delta v^3\left(\phi_L - v\right)$$

$$\Delta_{\rm max} = \frac{1}{3\sqrt{3}} \qquad \delta = \left[9\left(\sqrt{\Delta^2 - \Delta_{\rm max}^2} - \Delta\right)\right]^{1/3}$$

$$\varepsilon_\alpha = \frac{3^{1/3}\delta^2 - \delta^4 - 3^{2/3}}{2(3^{1/3} + \delta^2)^2} \in [0, 1]$$



$$V_C(\phi_C) = \frac{1}{2}m^2\phi_C^2 + \eta\phi_C^3 + \frac{1}{8}\lambda_C\phi_C^4$$

From Linear to Cubic (and back again)

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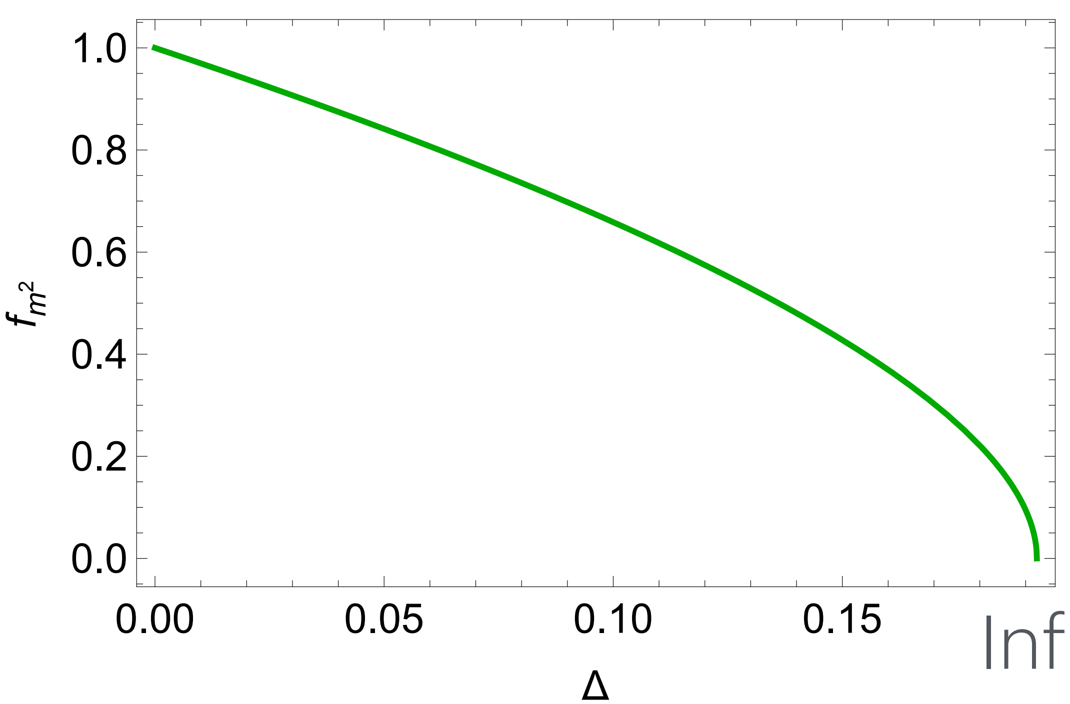
$$\varepsilon_\alpha = \frac{3^{1/3}\delta^2 - \delta^4 - 3^{2/3}}{2(3^{1/3}+\delta^2)^2}$$

$$m^2 = \lambda v^2 f_{m^2} \qquad f_{m^2} = \frac{1}{6}\left(3^{2/3}\delta^2 + \frac{3^{4/3}}{\delta^2} + 3\right)$$

$$\eta = \frac{\lambda v}{2} f_\eta$$



$m \to 0$

Inflection

$$V_C(\phi_C) = \frac{1}{2}m^2\phi_C^2 + \eta\phi_C^3 + \frac{1}{8}\lambda_C\phi_C^4$$

From Linear to Cubic (and back again)

Matteini, MN, Shoji, Ubaldi '24

$$L \to C$$

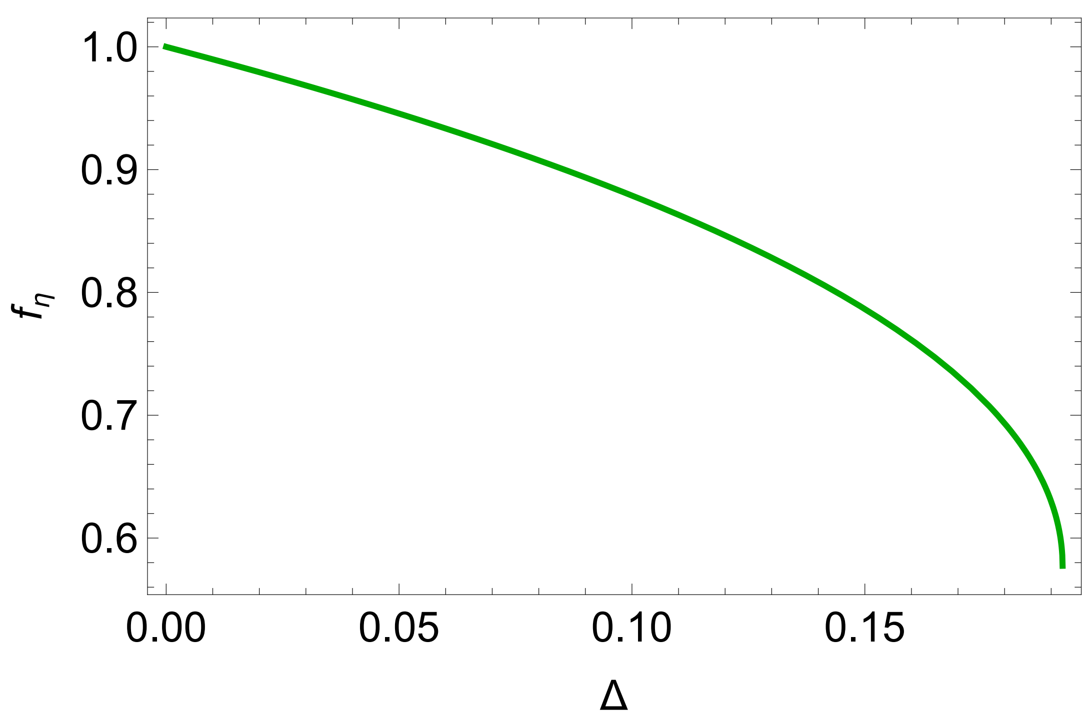
$$\{\lambda, v, \Delta\} \to \{\epsilon_\alpha, m, \eta\}$$

$$V_L(\phi_L) = \frac{\lambda}{8}\left(\phi_L^2 - v^2\right)^2 + \lambda\Delta v^3\left(\phi_L - v\right)$$

$$\varepsilon_\alpha = \frac{3^{1/3}\delta^2 - \delta^4 - 3^{2/3}}{2(3^{1/3}+\delta^2)^2}$$

$$m^2 = \lambda v^2 f_{m^2}$$

$$\eta = \frac{\lambda v}{2} f_\eta \qquad f_\eta = \frac{\delta^2 + 3^{1/3}}{3^{2/3}\delta}$$



$$V_C(\phi_C) = \frac{1}{2}m^2\phi_C^2 + \eta\phi_C^3 + \frac{1}{8}\lambda_C\phi_C^4$$

From Linear to Cubic (and back again)

Matteini, MN, Shoji, Ubaldi '24

$$L \to C$$

$$\{\lambda, v, \Delta\} \to \{\epsilon_\alpha, m, \eta\}$$

$$C \to L$$

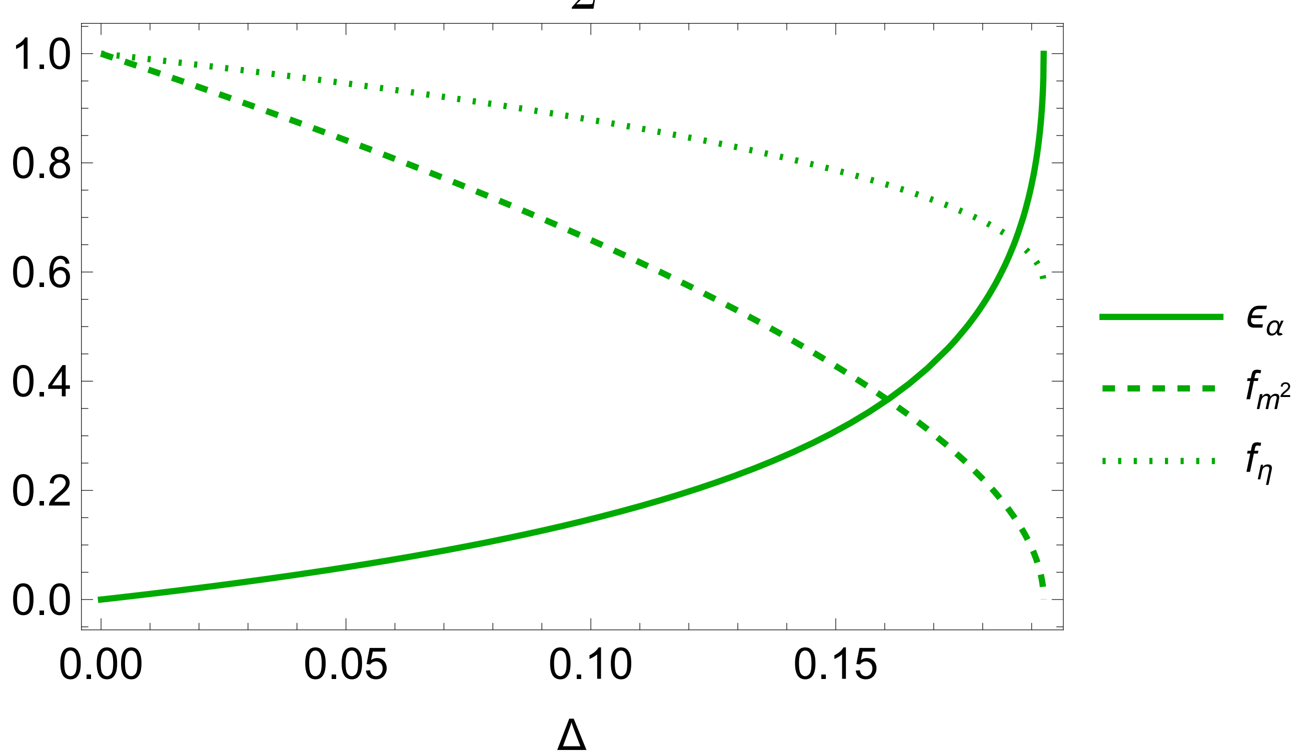
$$\{\epsilon_\alpha, m, \eta\} \to \{\lambda, v, \Delta\}$$

$$V_L(\phi_L) = \frac{\lambda}{8}\left(\phi_L^2 - v^2\right)^2 + \lambda\Delta v^3\left(\phi_L - v\right)$$

$$\varepsilon_\alpha = \frac{3^{1/3}\delta^2 - \delta^4 - 3^{2/3}}{2(3^{1/3}+\delta^2)^2}$$

$$m^2 = \lambda v^2 f_{m^2}$$

$$\eta = \frac{\lambda v}{2} f_\eta$$

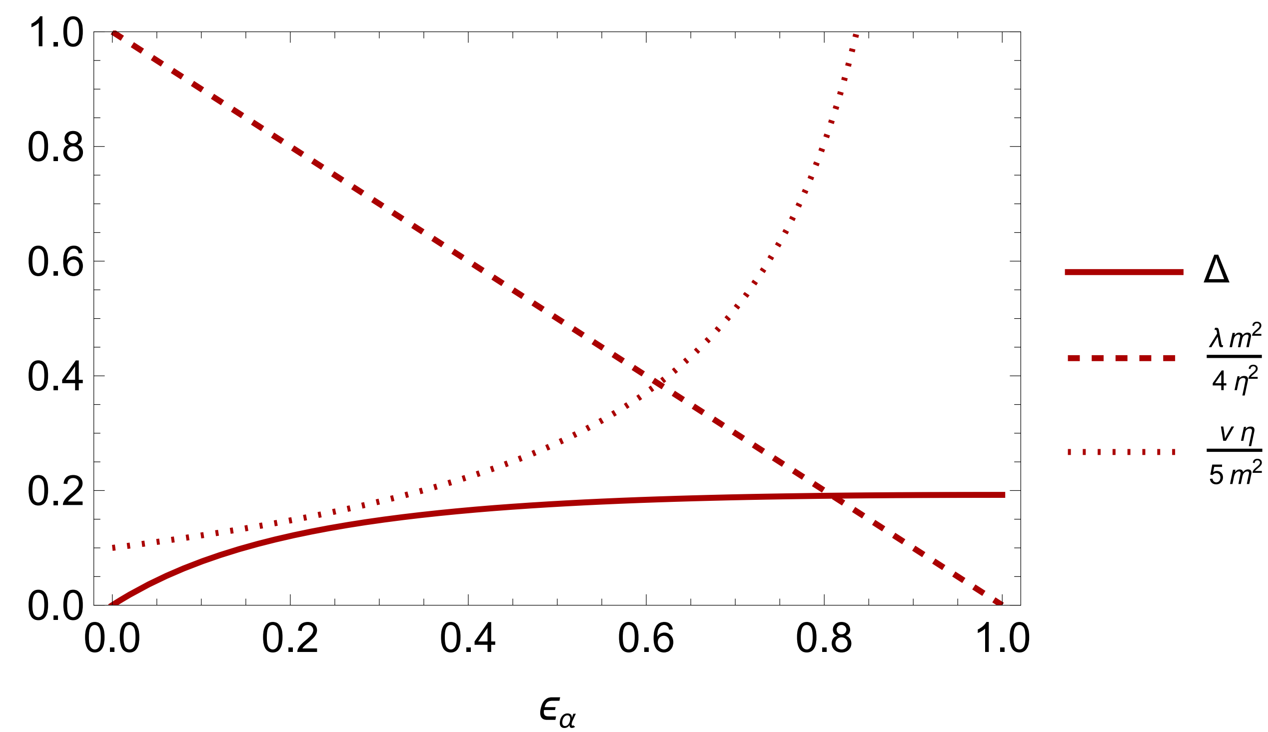


$$V_C(\phi_C) = \frac{1}{2}m^2\phi_C^2 + \eta\phi_C^3 + \frac{1}{8}\lambda_C\phi_C^4$$

$$\Delta = \frac{\varepsilon_\alpha}{(1+2\varepsilon_\alpha)^{3/2}}$$

$$\lambda = \frac{4\eta^2}{m^2}\left(1-\varepsilon_\alpha\right)$$

$$v = \frac{m^2}{2\eta}\frac{\sqrt{1+2\varepsilon_\alpha}}{1-\varepsilon_\alpha}$$



From Linear to Cubic (and back again)

Matteini, MN, Shoji, Ubaldi '24

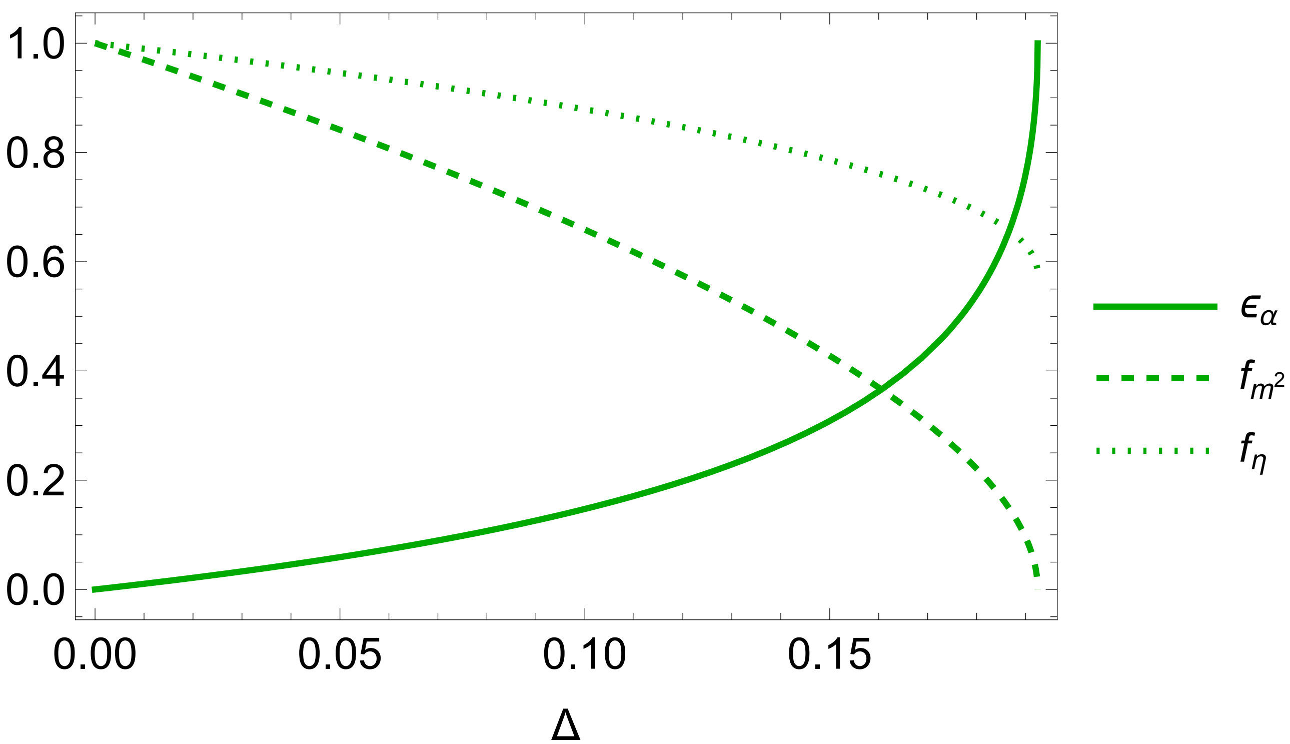
$L \to C$	$C \to L$
$\{\lambda, v, \Delta\} \to \{\epsilon_\alpha, m, \eta\}$	$\{\epsilon_\alpha, m, \eta\} \to \{\lambda, v, \Delta\}$

$$V_L(\phi_L) = \frac{\lambda}{8}\left(\phi_L^2 - v^2\right)^2 + \lambda \Delta v^3 \left(\phi_L - v\right)$$

$$\varepsilon_\alpha = \frac{3^{1/3}\delta^2 - \delta^4 - 3^{2/3}}{2(3^{1/3} + \delta^2)^2}$$

$$m^2 = \lambda v^2 f_{m^2}$$

$$\eta = \frac{\lambda v}{2} f_\eta$$



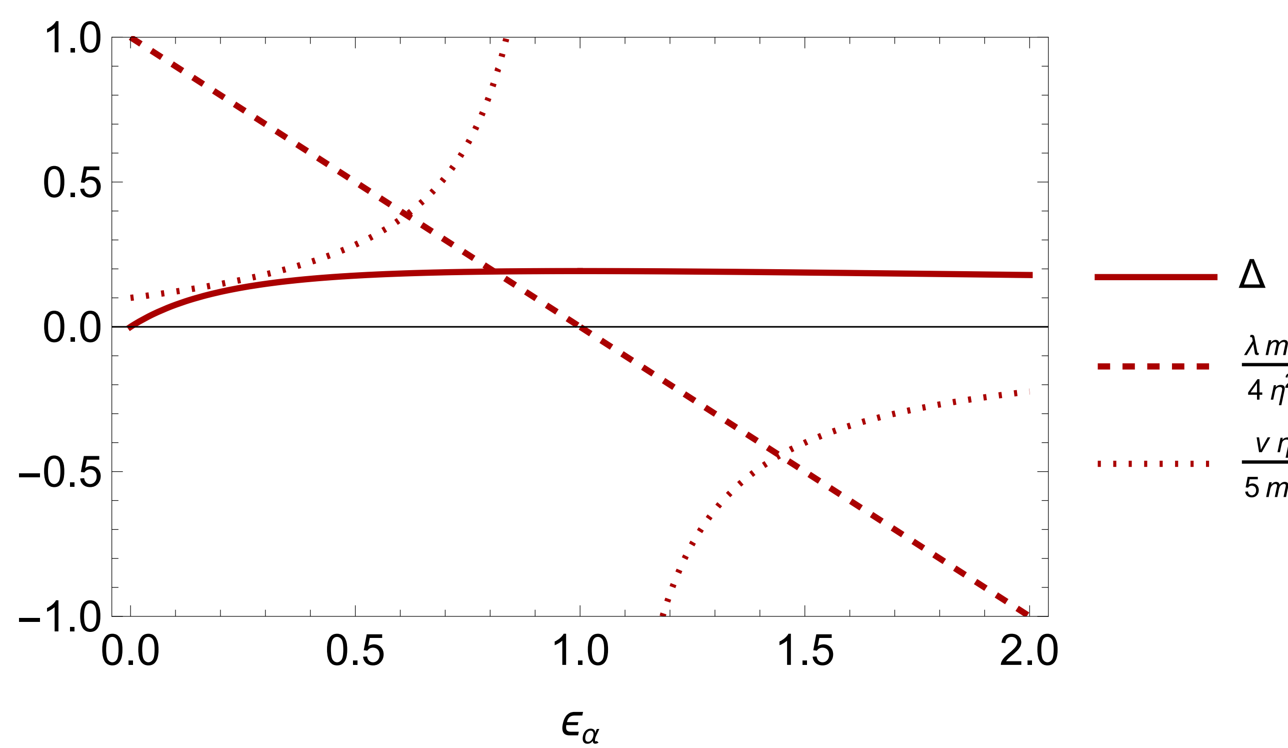
$$V_C(\phi_C) = \frac{1}{2}m^2\phi_C^2 + \eta\phi_C^3 + \frac{1}{8}\lambda_C\phi_C^4$$

$$\Delta = \frac{\varepsilon_\alpha}{(1 + 2\varepsilon_\alpha)^{3/2}}$$

$$\lambda = \frac{4\eta^2}{m^2}(1 - \varepsilon_\alpha)$$

$$v = \frac{m^2}{2\eta}\frac{\sqrt{1 + 2\varepsilon_\alpha}}{1 - \varepsilon_\alpha}$$

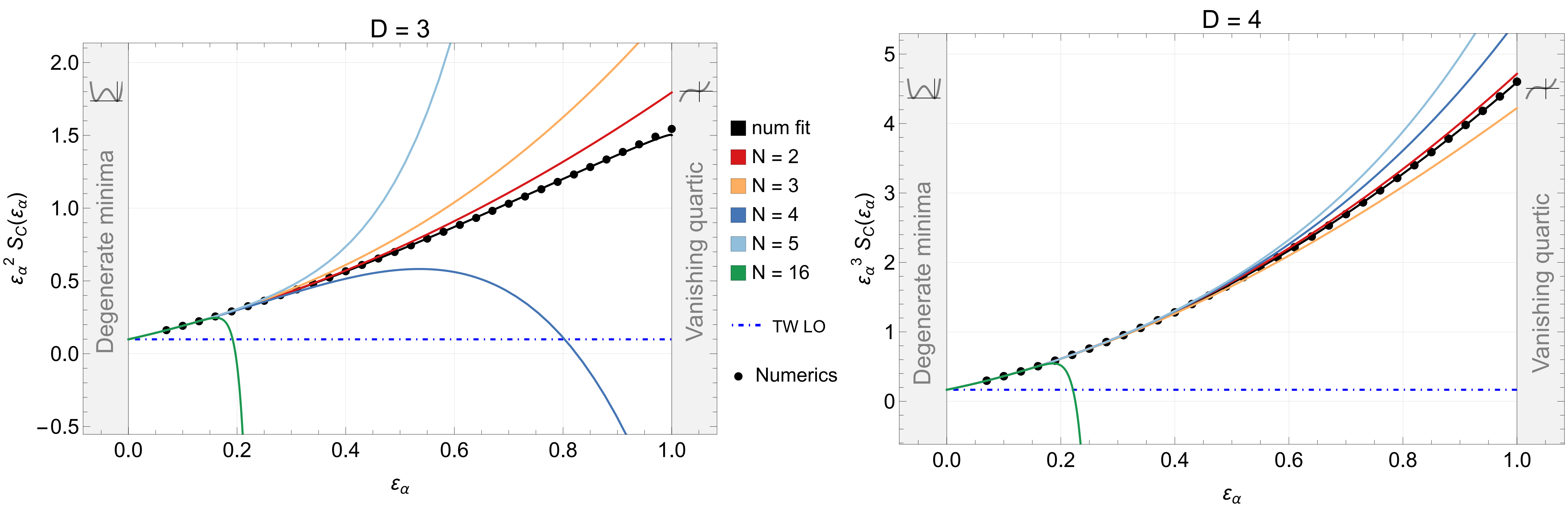
unstable



From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Cubic potential, optimal truncation at *N=2*



Hybrid approach, truncated at *N=2*, exactly translated to linear

$$S_L(\Delta) = \frac{\lambda^{D/2-1}}{v^{4-D}} \frac{m^{6-D}}{4\eta^2} S_C(\varepsilon_\alpha) \simeq \frac{f_{m^2}^{3-D/2}(\Delta)}{f_\eta^2(\Delta)} S_C^{(2)}(\varepsilon_\alpha(\Delta))$$

From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Hybrid approach: *cubic to linear*

$$\frac{v^{4-D}}{\lambda^{D/2-1}} S_L(\Delta) = \frac{m^{6-D}}{4\eta^2} S_C(\varepsilon_\alpha)$$

Exact, e.g. numerical

Approximations truncated, cubic optimal at *N*=2

$$S_L(\Delta) \simeq S_L(\Delta)^{(N)} \qquad S_C(\varepsilon_\alpha) \simeq S_C^{(N)}(\varepsilon_\alpha) \xrightarrow{\text{opt.}} S_C^{(2)}(\varepsilon_\alpha)$$

exactly translated to linear

sends S to zero at inflection

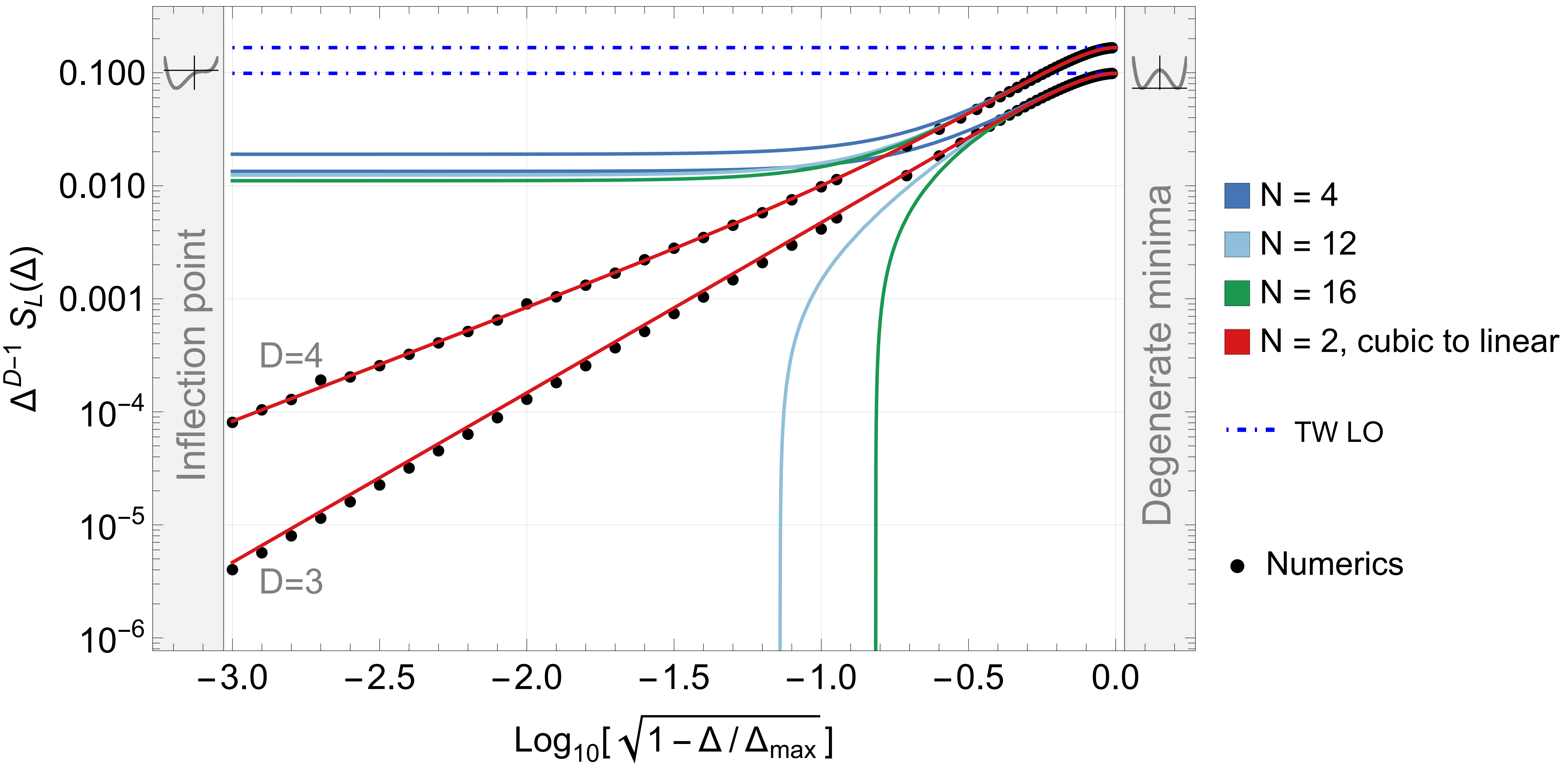
$$S_L(\Delta) = \frac{\lambda^{D/2-1}}{v^{4-D}} \frac{m^{6-D}}{4\eta^2} S_C(\varepsilon_\alpha) \simeq \frac{f_{m^2}^{3-D/2}(\Delta)}{f_\eta^2(\Delta)} S_C^{(2)}(\varepsilon_\alpha(\Delta))$$

exact map

From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Zoom into the inflection point, very flat potential, e.g. supercooled

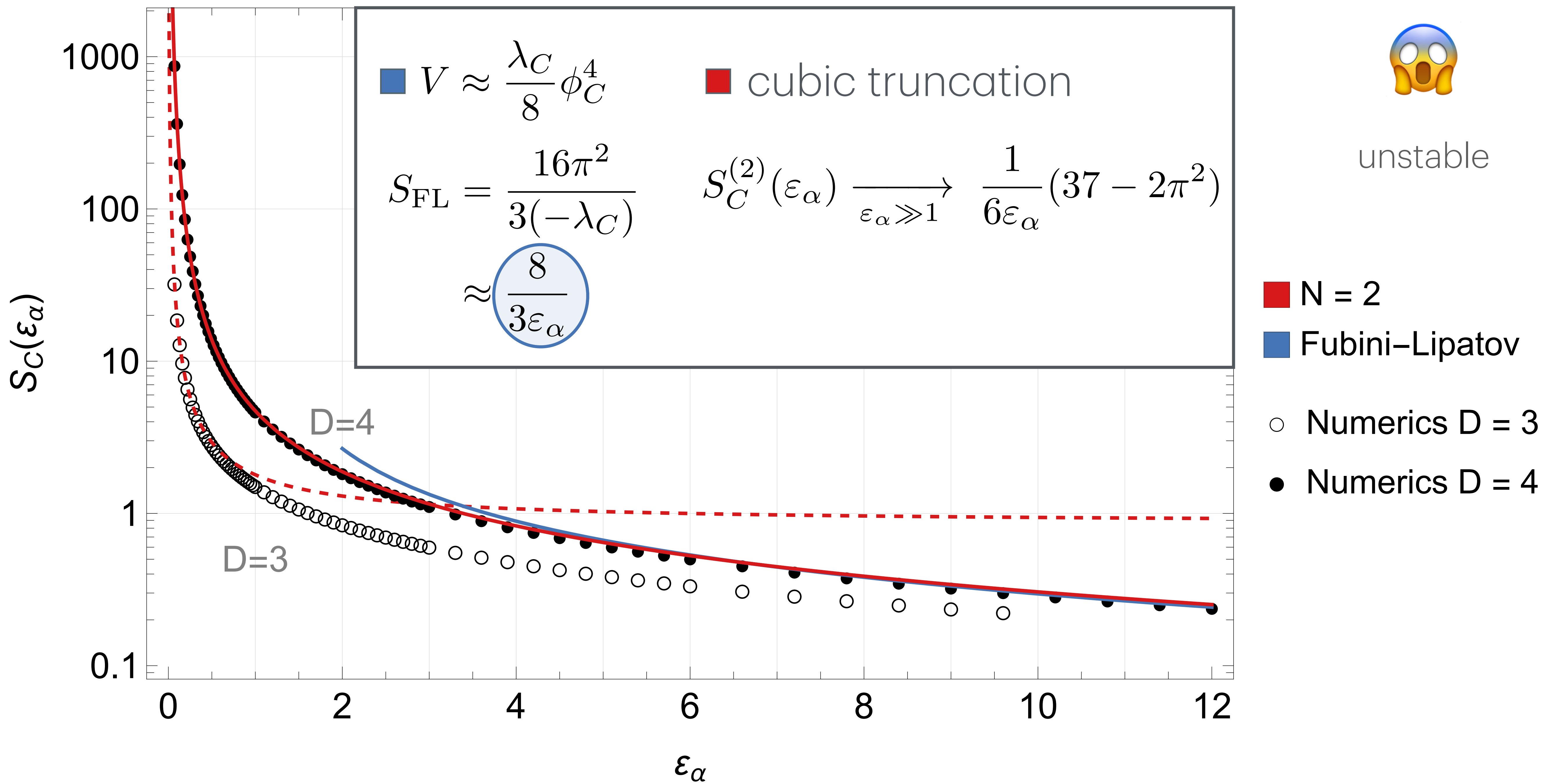


From Thin to Thick

Matteini, MN, Shoji, Ubaldi '24

Even unstable potentials, (a bit crazy since we started from TW)

unstable



Fluctuations

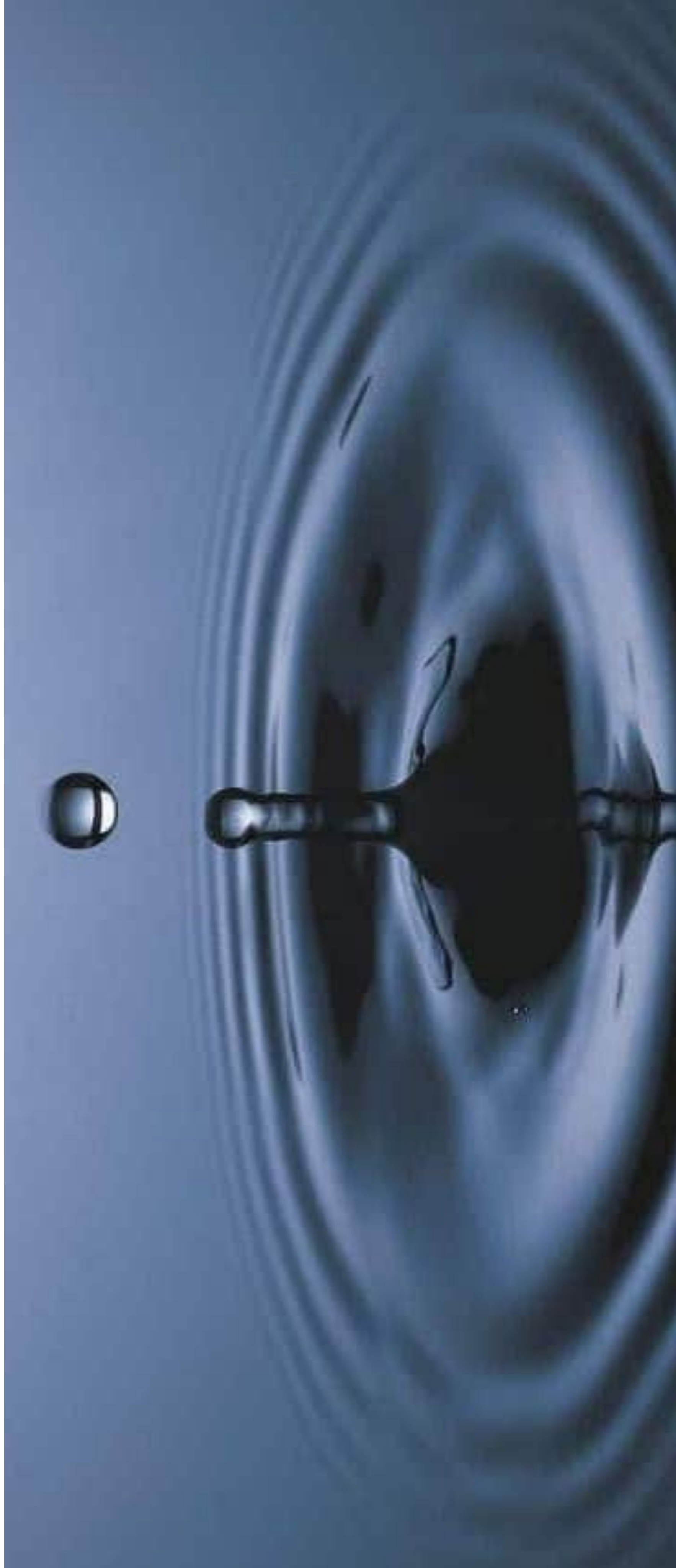
$$\int \mathcal{D}\varphi \to \prod \lambda_i$$

Thin wall — Ivanov, Matteini, MN, Ubaldi '22

Thin to Thick — Matteini, MN, Shoji, Ubaldi '24

* Standard Model Rate — Baratella, MN, Shoji, Trailović, Ubaldi '24

* not this talk, some aspects relevant here



Decay rate at one loop

Callan, Coleman '77

$$\gamma = \frac{\Gamma}{\mathcal{V}} = \left(\frac{S_R}{2\pi\hbar}\right)^{\frac{D}{2}} \left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S_{\rm ct}} \left(1 + \mathcal{O}(\hbar)\right)$$

$\mathcal{O} = -\partial_\mu \partial^\mu + V^{(2)}$ fluctuation operator around the bounce

Bubble deformation, zeroes for symmetries, single negative

Renormalized action, counter-terms $S_R, S_{\rm ct}$

Functional determinants $\det \mathcal{O}$

Lecture notes by

Coleman Erice '88

Dunne '07

Zero removal $\det' \mathcal{O}$

Renormalized bounce action, counter-terms

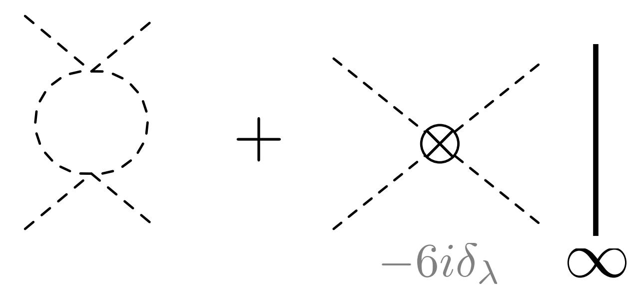
Ivanov, Matteini, MN, Ubaldi '22

One loop counter-terms

Peskin & Schröder

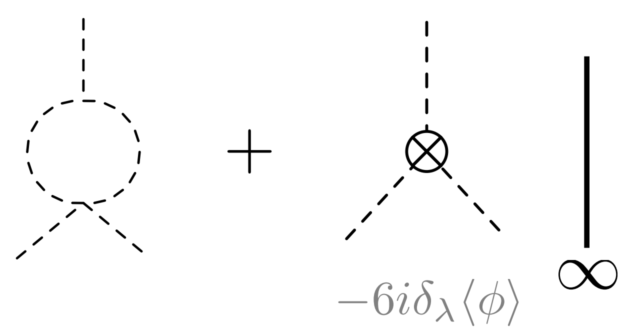
$$V_{\rm ct} = \frac{\delta_{m^2}}{2}\phi^2 + \frac{\delta_\lambda}{4}\phi^4, \qquad \langle\phi\rangle = \frac{\mu}{\sqrt{\lambda}}, \qquad V^{(n)} \equiv \frac{\mathrm{d}^n V}{\mathrm{d}\phi^n}(\langle\phi\rangle)$$

$$\delta_\lambda = \frac{1}{32\pi^2\varepsilon}V^{(4)2}$$ set by the 4-p function



$$= 0 \qquad \varepsilon = 4 - D$$

cancels the 3-p

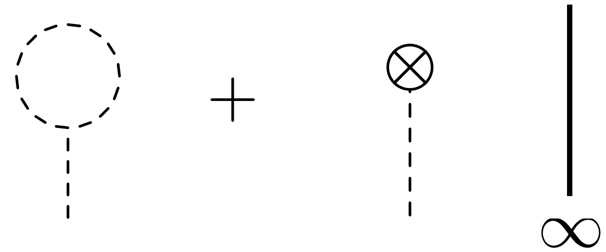


$$= 0$$ if $$V^{(3)} = V^{(4)}\langle\phi\rangle$$

automatic if 3-p interactions comes from a single quartic

$$V^{(4)} = 4!\,\lambda, \quad V^{(3)} = 4\times 3!\,\lambda\langle\phi\rangle = V^{(4)}\langle\phi\rangle$$

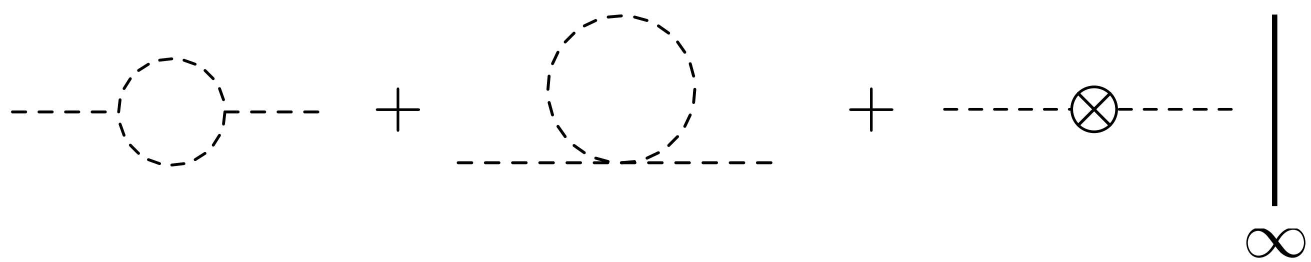
Remove the tadpoles...



$= 0$

$$\delta_{m^2} = \frac{1}{(4\pi)^2 \varepsilon} V^{(4)} \left(V^{(2)} - \frac{1}{2} V^{(4)} \langle \phi \rangle^2 \right)$$

...and the 2-p mass divergence cancels out



$= 0$

$$\langle \phi \rangle \simeq v \left(1 - \Delta + \ldots\right)$$

the vev shifts

$$\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2 \, 2\varepsilon} \,, \qquad \delta_{m^2} = -\frac{3\lambda^2 v^2}{(4\pi)^2 \, 2\varepsilon}$$

independent of Δ

Δ counter-term is zero, and does not run $\delta_\Delta = 0$

Renormalized bounce action, counter-terms

Counter-term for the Euclidean action

$$S_{\rm ct} = \int_D \left(V_{\rm ct} - V_{\rm ctFV}\right) = \frac{3\lambda^2}{8\left(4\pi\right)^2 \varepsilon} \int_D \left(3\left(\phi^4 - \phi_{\rm FV}^4\right) - 2v^2\left(\phi^2 - \phi_{\rm FV}^2\right)\right) \simeq -\frac{3}{16\varepsilon\Delta^3} \, .$$

Running of the Euclidean action

$$S_R + S_{\rm ct} = S\left(1 - \frac{9\lambda_0}{\left(4\pi\right)^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu}{\mu_0}\right)\right)$$

Non-trival check: the $\frac{1}{\varepsilon}$ pole and $\ln\mu$ cancel with the determinant

Functional determinants

We wish to get the eigenvalues of $\mathcal{O}$ and multiply them

The bounce and fluctuations spherically symmetric, orbital decomposition

$$\left|\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right|^{-\frac{1}{2}} = \left|\prod_{l=0}^{\infty} \frac{\det' \mathcal{O}_l}{\det \mathcal{O}_{l\rm FV}}\right|^{-\frac{1}{2}}$$

$$\mathcal{O}_l = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{D-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{l\,(l+D-2)}{\rho^2} + V^{(2)}\,, \qquad V^{(2)} = \left.\frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2}\right|_{\bar\varphi}$$

Gel'fand-Yaglom theorem

$$\mathcal{O}_l\,\psi_l = 0\,, \quad \psi_l(0) \sim \rho^l$$
$$\mathcal{O}_{l\rm FV}\,\psi_{l\rm FV} = 0\,, \quad \psi_{l\rm FV}(0) \sim \rho^l$$

$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\rm FV}} = \left(\left.\frac{\psi_l}{\psi_{l\rm FV}}\right|_{\infty}\right)^{d_l}$$

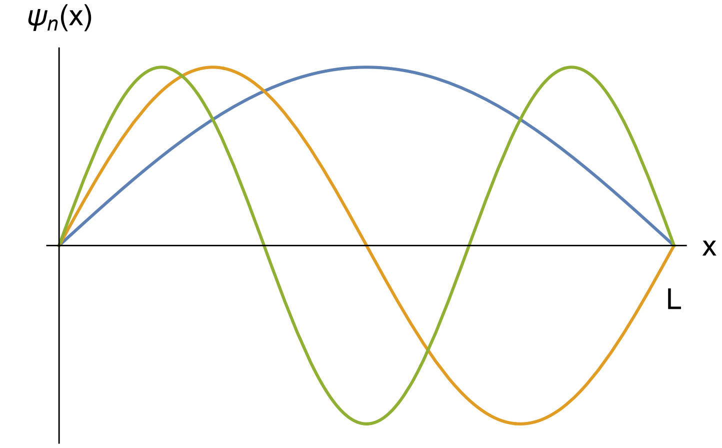
Scalar degeneracy factor $\quad d_l = \dfrac{(2l+D-2)(l+D-3)!}{l!(D-2)!}$

$$d_0 = 1$$
$$d_1 = D$$

Gel'fand-Yaglom theorem

simplest instance of GY formalism 'magic'

QM potential well, classical 1D string



$$\mathcal{O} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2 \qquad \mathcal{O}_{\mathrm{FV}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

a) impose Dirichlet (fixed) boundary condition at L

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 + m^2\,, \quad \lambda_{n\mathrm{FV}} = \left(\frac{n\pi}{L}\right)^2\,, \quad \frac{\det\,\mathcal{O}}{\det\,\mathcal{O}_{\mathrm{FV}}} = \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_{n\mathrm{FV}}} = \frac{\sinh(mL)}{mL}$$

b) solve the Cauchy (open) boundary condition

$$\psi'' - m^2\psi = 0, \quad \psi(0) = 0, \quad \psi'(0) = 1, \quad \psi = \frac{\sinh(mx)}{m}, \quad \psi_{\mathrm{FV}} = x$$

$$\frac{\det\,\mathcal{O}}{\det\,\mathcal{O}_{\mathrm{FV}}} = \frac{\psi}{\psi_{\mathrm{FV}}}(L) = \frac{\sinh(mL)}{mL}$$

Fluctuations

Ivanov, Matteini, MN, Ubaldi '22

Define the ratio $R_l \equiv \frac{\psi_l}{\psi_{l\mathrm{FV}}}$ and solve the stable GY equation

$$\ddot{R}_l + 2\left(\frac{\dot{\psi}_{l\mathrm{FV}}}{\psi_{l\mathrm{FV}}}\right)\dot{R}_l = \left(V^{(2)} - V^{(2)}_{\mathrm{FV}}\right) R_l$$

Baacke, Lavrelashvili '03

$$\psi_{l\mathrm{FV}}(0) \sim \rho^l\,, \qquad R_l(0) = 1\,, \qquad \dot{R}_l(0) = 0$$

IR $l \sim 1$ — l — UV $l \gg 1$

Low multipoles $l < \frac{1}{\Delta}$

Multiplicative TW expansion $\quad R_l = \prod_{n\geq 0} R_{ln}^{\Delta^n} \qquad x = e^z$

$$R_{l0} = \frac{1}{(1+x)^2}\,,$$

$$\ln R_{l1} = 3\left(r + \ln x\right)\,,$$

$$\ln R_{l2}(x\to\infty) = \frac{3}{4}\frac{(l-1)(l+D-1)}{(D-1)^2}x^2$$

$$R_l(\infty) = \Delta^2 e^{D-1}\frac{3}{4}\frac{(l-1)(l+D-1)}{(D-1)^2}$$

Negative and zero modes ok

IR $l \sim 1$ l UV $l \gg 1$

High multipoles $l \gg \frac{1}{\Delta}$

Shift multipoles $\nu = l + \frac{D}{2} - 1$

FV part $\psi_{\nu \mathrm{FV}} \simeq e^{k_\nu z}\,,$ $k_\nu^2 = 1 + \frac{\Delta^2 \nu^2}{r_0^2}$

Leading order $R_{\nu 0}(\infty) = \frac{(k_\nu - 1)\,(2k_\nu - 1)}{(k_\nu + 1)\,(2k_\nu + 1)}$

$$\ln R_\nu(\infty) = \ln \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1}\right)$$

Ivanov, Matteini, MN, Ubaldi '22

'All-order' corrections, long appendix

Correct UV behaviour $\lim_{\nu \to \infty} R_\nu(\infty) = 1$ not low-l, though, as expected

Fluctuations: summary

Ivanov, Matteini, MN, Ubaldi '22

IR $l \sim 1$ l UV $l \gg 1$

Low multipoles $\quad l < \dfrac{1}{\Delta}: \quad R_l(\infty) = \Delta^2 e^{D-1} \dfrac{3}{4} \dfrac{(l-1)(l+D-1)}{(D-1)^2}$ Negative, zero

High multipoles $\quad l \gg \dfrac{1}{\Delta}: \quad \ln R_\nu(\infty) = \ln \dfrac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1}\right)$

$$k_\nu^2 = 1 + \frac{\Delta^2 \nu^2}{r_0^2}$$

needs R_ν

Renormalized determinant

$$\sum_{\text{fin}} = \sum_{l} - \sum_{\text{asymp}} + \sum_{\text{ren}}$$

Two approaches (different power counting)

1a) organize in multipoles, minimal subtraction $\quad \sum_{\text{asymp}} \sim \#_1 \nu + \#_2 \frac{1}{\nu}$

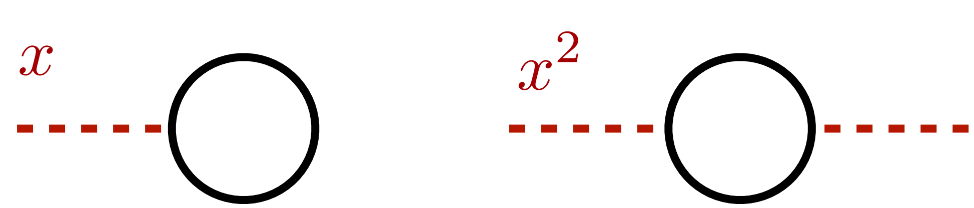
quadratic log divergence

1b) organize in coupling x insertion $\quad \sum_{\text{asymp}} \sim \#_1 x + \tilde{\#}_2 x^2$

Renormalize - replace divergencies with $\overline{\text{MS}}$ and introduce μ

2a) ζ function

2b) Feynman diagrams



$$\sum_{\text{ren}} \propto \frac{1}{\varepsilon} + \ln \mu + \text{fin}$$

Renormalized determinant

Sum over multipoles $$\ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\rm FV}}\right) = \sum_{\nu=D/2-1}^{\infty} d_\nu \ln R_\nu \,, \qquad d_\nu \simeq \frac{2}{(D-2)!}\nu^{D-2}$$

Diverges in the UV as expected from QFT

$$\sum_{\nu\gg 1} d_\nu \ln R_{\nu\gg 1} \sim -\frac{3r_0(2-r_0)}{(D-2)!\Delta}\sum_{\nu\gg 1}\nu^{D-2}\left(\frac{1}{\nu}-\frac{1}{\nu^3}\left(\frac{r_0}{2\Delta}\right)^2\right)$$

quadratic and log in *D=4*

Finite sum $$\Sigma_D = \sum_{\nu=\nu_0}^{\infty}\sigma_D = \sum_{\nu=\nu_0}^{\infty} d_\nu\left(\ln R_\nu - \ln R_\nu^a\right)$$

asymptotic subtraction

Renormalized determinants
(subtractions and logs for Ds)

WKB — Dunne, Min '05

ζ — Dunne, Kirsten '06

Improved — Hur, Min '08

Renormalized determinant *D=4*

$$\ln\left(\frac{\det\mathcal{O}}{\det\mathcal{O}_{\rm FV}}\right) = \sum_\nu \nu^2\left(\ln R_\nu - \frac{1}{2\nu}I_1 + \frac{1}{8\nu^3}I_2\right) - \frac{1}{8}\tilde{I}_2$$

Asymptotic subtractions remove first two divergencies in any *D*

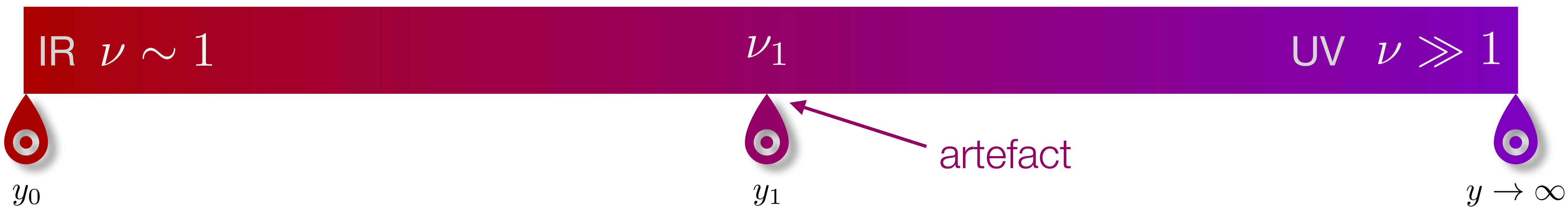
$$I_1 = \int_0^\infty {\rm d}\rho\,\rho\left(V^{(2)} - V^{(2)}_{\rm FV}\right) \simeq -3\,(2-r_0)\left(\frac{r_0}{\Delta}\right),$$
$$I_2 = \int_0^\infty {\rm d}\rho\,\rho^3\left(V^{(2)2} - V^{(2)2}_{\rm FV}\right) \simeq -3\,(2-r_0)\left(\frac{r_0}{\Delta}\right)^3$$

The renormalized piece gives the pole, the log scale and a finite piece

$$\tilde{I}_2 = \int_0^\infty {\rm d}\rho\,\rho^3\left(V^{(2)2} - V^{(2)2}_{\rm FV}\right)\left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln\left(\frac{\mu\rho}{2}\right)\right)$$
$$\simeq I_2\left(\frac{1}{\varepsilon} + \gamma_E + \frac{5}{4} + \ln\left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta}\right)\right)$$

Final sum done by Euler-Maclaurin, dominated by large multipoles, use $y = \frac{\Delta\nu}{r_0}$

$$\Sigma_4^{\int} \simeq \frac{1}{\Delta^3}\int_{y_0}^{\infty} \mathrm{d}y\, y^2 \left(\ln R_\nu + \frac{3}{2y} - \frac{3}{8y^3}\right) = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} + \ln 2y_0\right)$$



Separate low and high at arbitrary intermediate multipole ν_1

$$\Sigma_D = \Sigma_D^{\text{low}} + \Sigma_D^{\text{high}} = \sum_{\nu=\nu_0}^{\nu_1} \sigma_D + \sum_{\nu=\nu_1+1}^{\infty} \sigma_D$$

Combining low and high, we get the finite sum

$$\Sigma_4 = \frac{3}{8\Delta^3}\left(\frac{9-4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta\right) \quad \Rightarrow \quad \ln\left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}}\right) = \Sigma_4 - \frac{\tilde{I}_2}{8}$$

The γ_E and $\ln\Delta$ cancel with parts in $\tilde{I}_2 \simeq I_2\left(\frac{1}{\varepsilon} + \gamma_E + \frac{5}{4} + \ln\left(\frac{\mu r_0}{2\sqrt{\lambda}v\Delta}\right)\right)$

Zero removal

$$\Gamma \propto \frac{1}{\sqrt{\det \mathcal{O}'}} = \prod_n \sqrt{\frac{\lambda_{n\mathrm{FV}}}{\lambda'_n}} \propto (v^2)^{D/2}$$

Eigenvalues have d=2, drop *D* of them, sqrt overall

With Gelfand-Yaglom, all the *l=1* eigenvalues are multiplied

Modify GY

$$\left(\mathcal{O}_{l=1} + \mu_\varepsilon^2\right) \psi_{l=1}^{\varepsilon} = 0$$

$$R_{l=1}^{\varepsilon}(\infty) = \frac{\psi_{l=1}^{\varepsilon}(\infty)}{\psi_{l=1}^{\mathrm{FV}}(\infty)} \simeq \frac{\left(\mu_\varepsilon^2 + \gamma_1\right) \prod_{n=2}^{\infty} \gamma_n}{\prod_{n=1}^{\infty} \gamma_n^{\mathrm{FV}}} = \mu_\varepsilon^2 R'_{l=1}(\infty)$$

We already have the fluctuation functions, easy to off-set

$$R'_{l=1}(\infty) = \lim_{\mu_\varepsilon^2 \to 0} \frac{1}{\mu_\varepsilon^2} R_{l=1}^{\varepsilon}(\infty) = \frac{e^{D-1}}{12} \frac{1}{\lambda v^2}$$

* two other ways of seeing the same thing give the same answer

This answers the question of dimensional analysis estimate

TW rate at one loop - summary

Ivanov, Matteini, MN, Ubaldi '22

Explicit closed form renormalized rate at one loop $\mu_0 = \sqrt{\lambda} v$

$$\frac{\Gamma}{\mathcal{V}} \simeq \left(\left(\frac{S}{2\pi}\right)\frac{12}{e^{D-1}}\lambda v^2\right)^{D/2} \exp\left[-S - \frac{1}{\Delta^{D-1}}\begin{cases}\frac{20+9\ln 3}{54}, & D=3\,,\\ \frac{27-2\pi\sqrt{3}}{96}, & D=4\,,\end{cases}\right]$$

zero removal

determinant part

Renormalized action

$$S = \frac{1}{\Delta^{D-1}}\begin{cases}\frac{2^5\pi v}{3^4\sqrt{\lambda}}\left(1-\left(\frac{9\pi^2}{4}-1\right)\Delta^2\right), & D=3\\ \frac{\pi^2}{3\lambda}\left(1-\left(2\pi^2+\frac{9}{2}\right)\Delta^2\right), & D=4\end{cases}$$

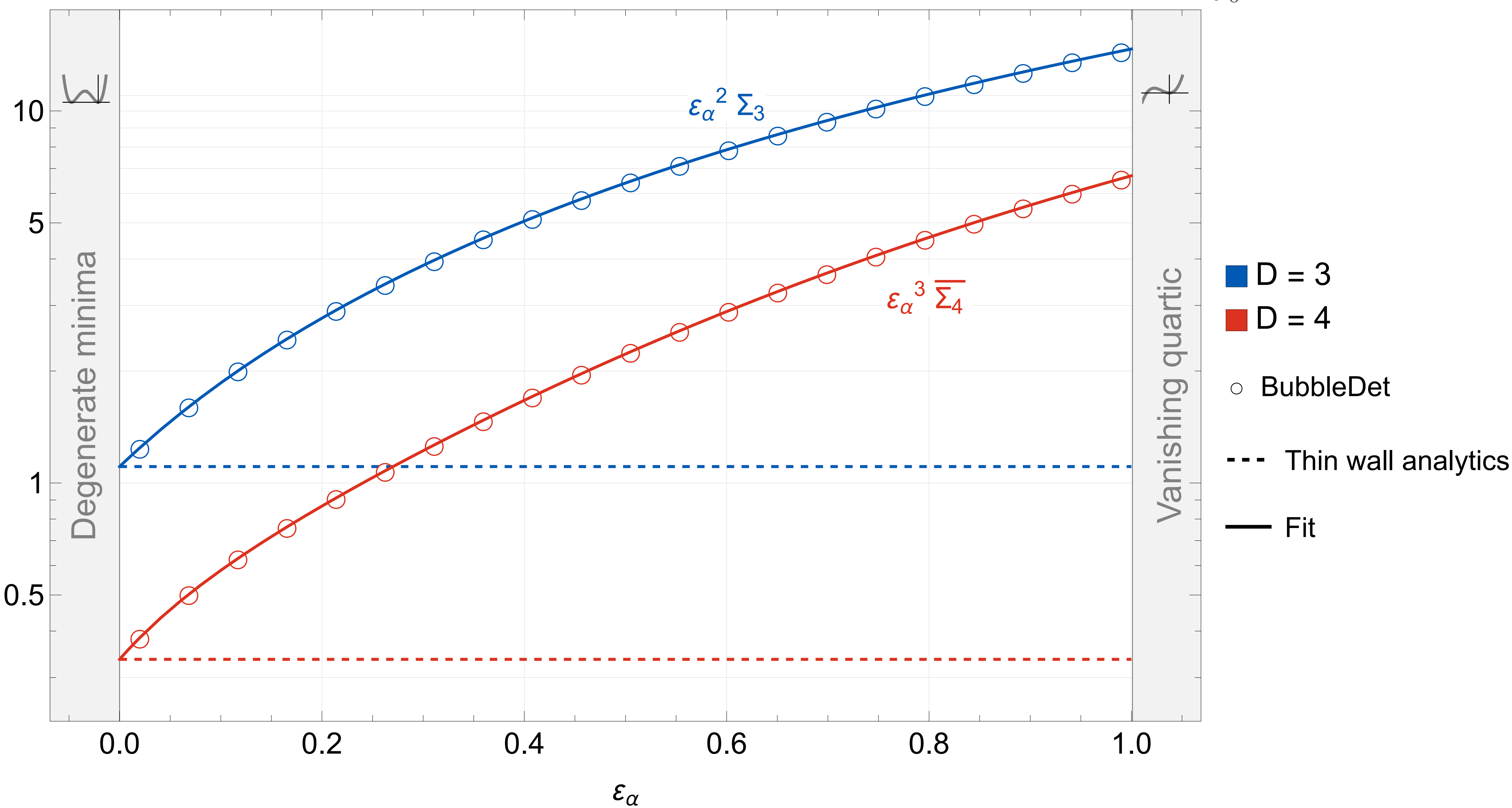
General procedure for even and odd *D*

Renormalized sums

$$\Sigma_3 = \sum_{\nu=1/2} 2\nu \left(\ln R_\nu - \frac{1}{2\nu} I_1 \right)$$

Matteini, MN, Shoji, Ubaldi '24

$$I_1 = \int_0^\infty \mathrm{d}\rho\, \rho \left(V^{(2)} - V^{(2)}_{\mathrm{FV}} \right)$$

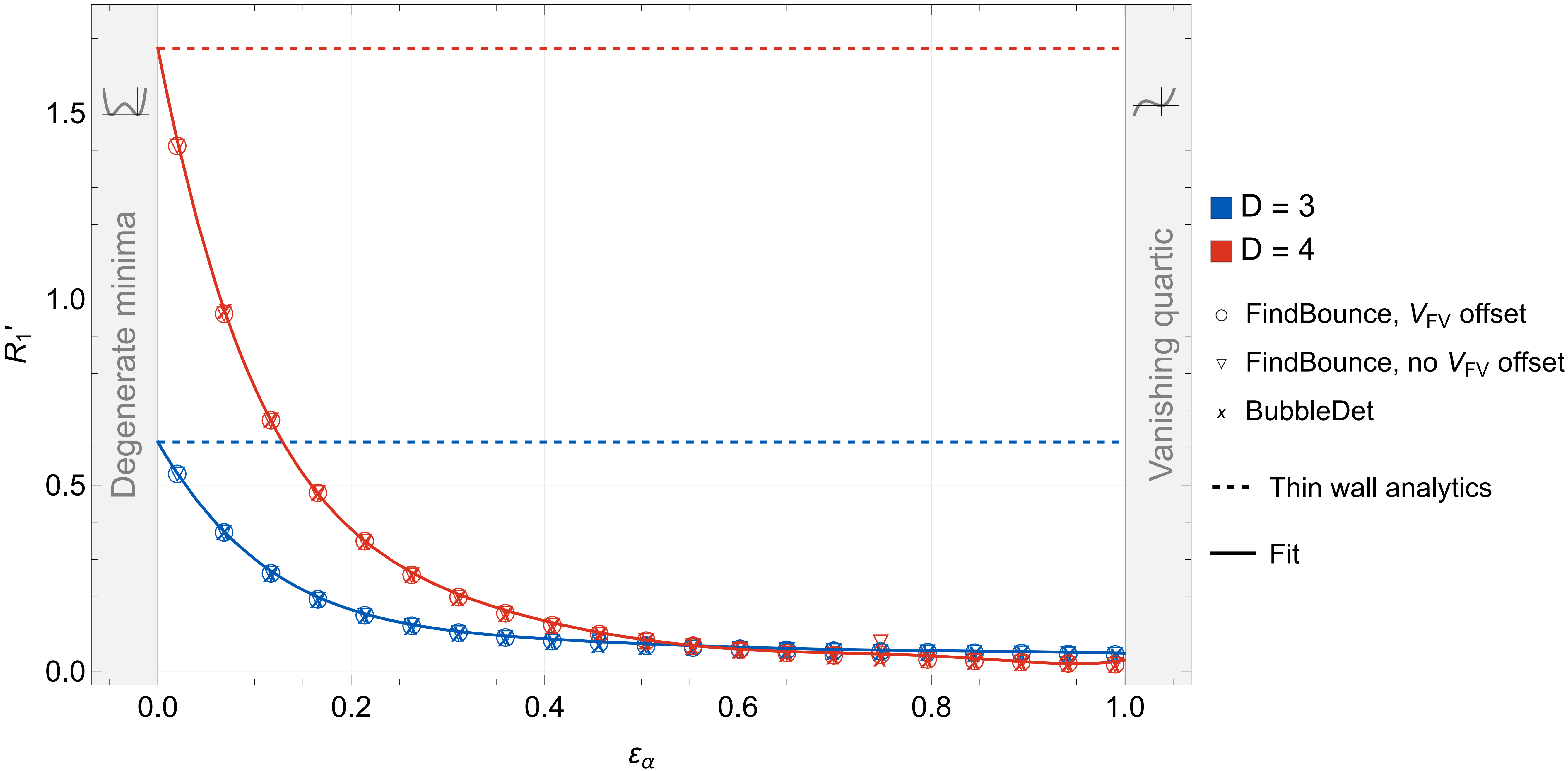


Simple fit

$$\Sigma_3(\varepsilon_\alpha) = \frac{20 + 9\ln 3}{27} \frac{1}{\varepsilon_\alpha^2} \left(1 + 6.0\varepsilon_\alpha + 8.0\varepsilon_\alpha^2 - 1.8\varepsilon_\alpha^3\right)$$

Zero removal

Matteini, MN, Shoji, Ubaldi '24



Thin wall: $R_1' = \dfrac{e^{D-1}}{12}$

Polynomial fit $R_1'(\varepsilon_\alpha) = \dfrac{e^2}{12}\left(1 - 7.3\varepsilon_\alpha + 27\varepsilon_\alpha^2 + \dots\right)$

Multiple methods, same result, similar for the linear potential

Summary

Matteini, MN, Shoji, Ubaldi '24

Convenient cubic

$$V = \frac{1}{2} m^2 \phi^2 + \eta \phi^3 + \frac{\lambda}{8} \phi^4 \,, \qquad \epsilon_\alpha \equiv 1 - \lambda \frac{m^2}{4\eta^2}$$

Rate

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{S_D}{2\pi} \right)^{D/2} m^D e^{-S_D - \frac{1}{2}\Sigma_D}$$

Full One Loop, from thin to thick (inflection)

$$S_3 = \frac{32\pi}{81} \frac{m^3}{4\eta^2} \frac{1}{\varepsilon_\alpha^2} \left[1 + \frac{17}{2} \varepsilon_\alpha + \left(\frac{247}{8} - \frac{9\pi^2}{4} \right) \varepsilon_\alpha^2 \right]$$

$$S_4 = \frac{\pi^2}{3} \frac{m^2}{4\eta^2} \frac{1}{\varepsilon_\alpha^3} \left[1 + 10\varepsilon_\alpha + \left(37 - 2\pi^2 \right) \varepsilon_\alpha^2 \right]$$

$$\Sigma_3 = \frac{20 + 9\ln 3}{27} \frac{1}{\varepsilon_\alpha^2} \left(1 + 6.0\varepsilon_\alpha + 8.0\varepsilon_\alpha^2 - 1.8\varepsilon_\alpha^3 \right)$$

$$\Sigma_4 = \frac{27 - 2\pi\sqrt{3}}{48} \frac{1}{\varepsilon_\alpha^3} \left(1 + 7.2\varepsilon_\alpha - 0.6\varepsilon_\alpha^2 + 24\varepsilon_\alpha^3 - 15\varepsilon_\alpha^4 + 3.5\varepsilon_\alpha^5 \right)$$

Tunneling in QFT

11 January 2024 to 31 December 2030
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andreas.ekstedt@physi…
oliver.gould@nottingha…
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Seminar series devoted to tunneling in QFT

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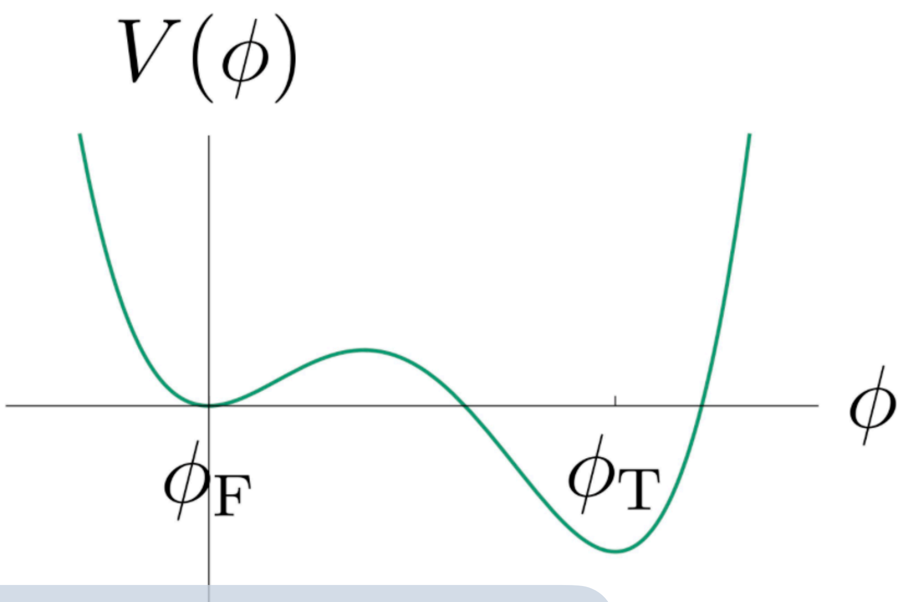

Indico: https://indico.global/event/5685

Please register on this Indico page to receive Zoom joining instructions plus a reminder for each talk.

Youtube: https://www.youtube.com/@TunnelingQFT