UNIVERSITY OF LJUBLJANA
FMF
Faculty of Mathematics and Physics
IJS

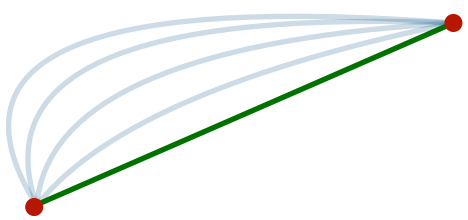
From Polygonal Bounces to FindBounce

Joint Yonsei-Konkuk-Sogang Mini-workshop on FOPT

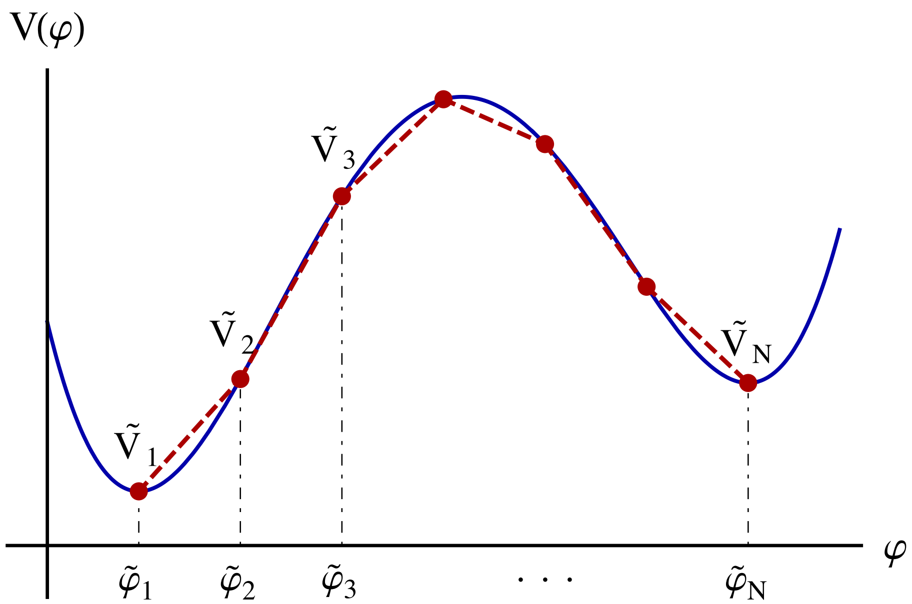
Miha Nemevšek, 6 February 2025

Slovenian Research and Innovation Agency

Outline

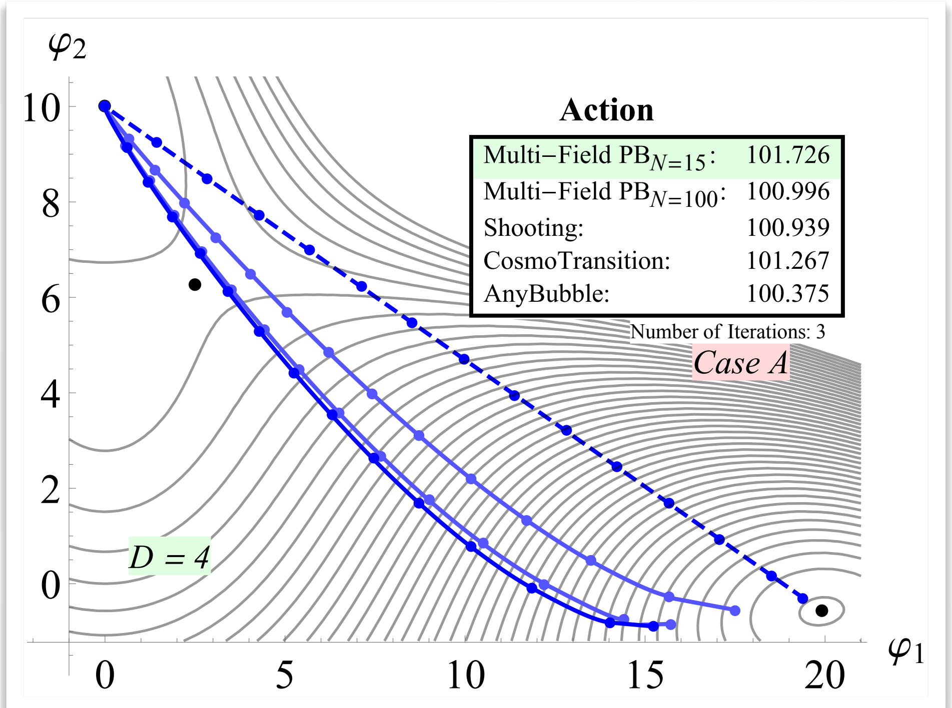


$$\delta S = 0 \rightarrow \overline{\varphi}(\rho)$$



Polygonal bounces

Guada, Maiezza, MN '18



FindBounce

Guada, MN, Pintar '20

Bounce

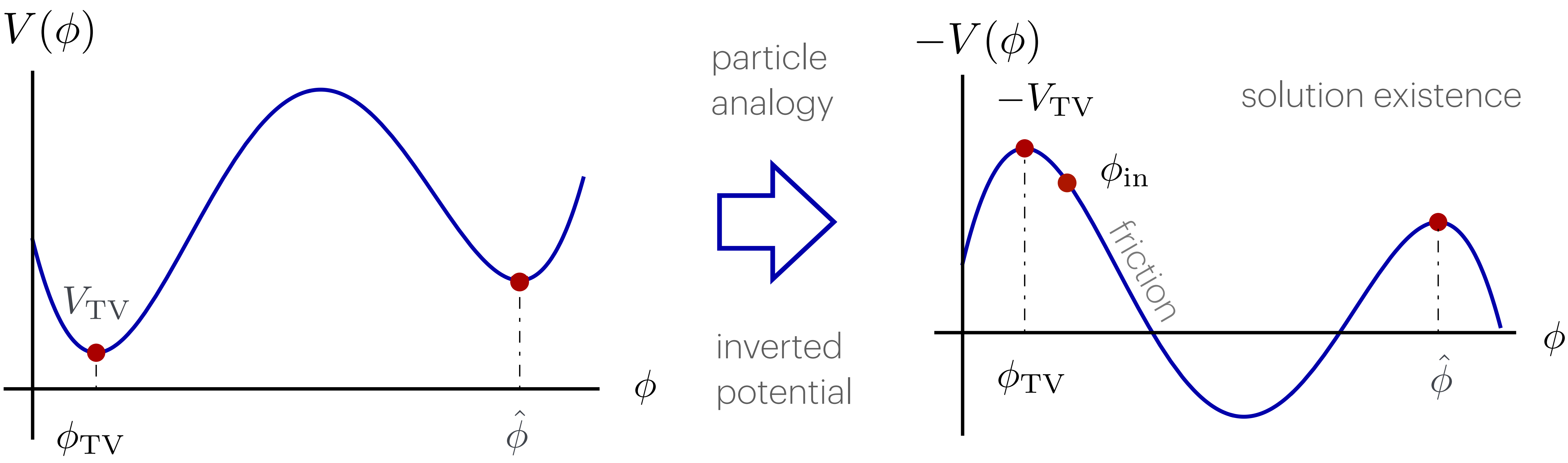
Coleman '77

Action $$S = \Omega \int_0^\infty \mathrm{d}\rho\, \rho^{D-1} \left(\frac{1}{2}\dot{\phi}^2 + V - \hat{V} \right) , \qquad \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

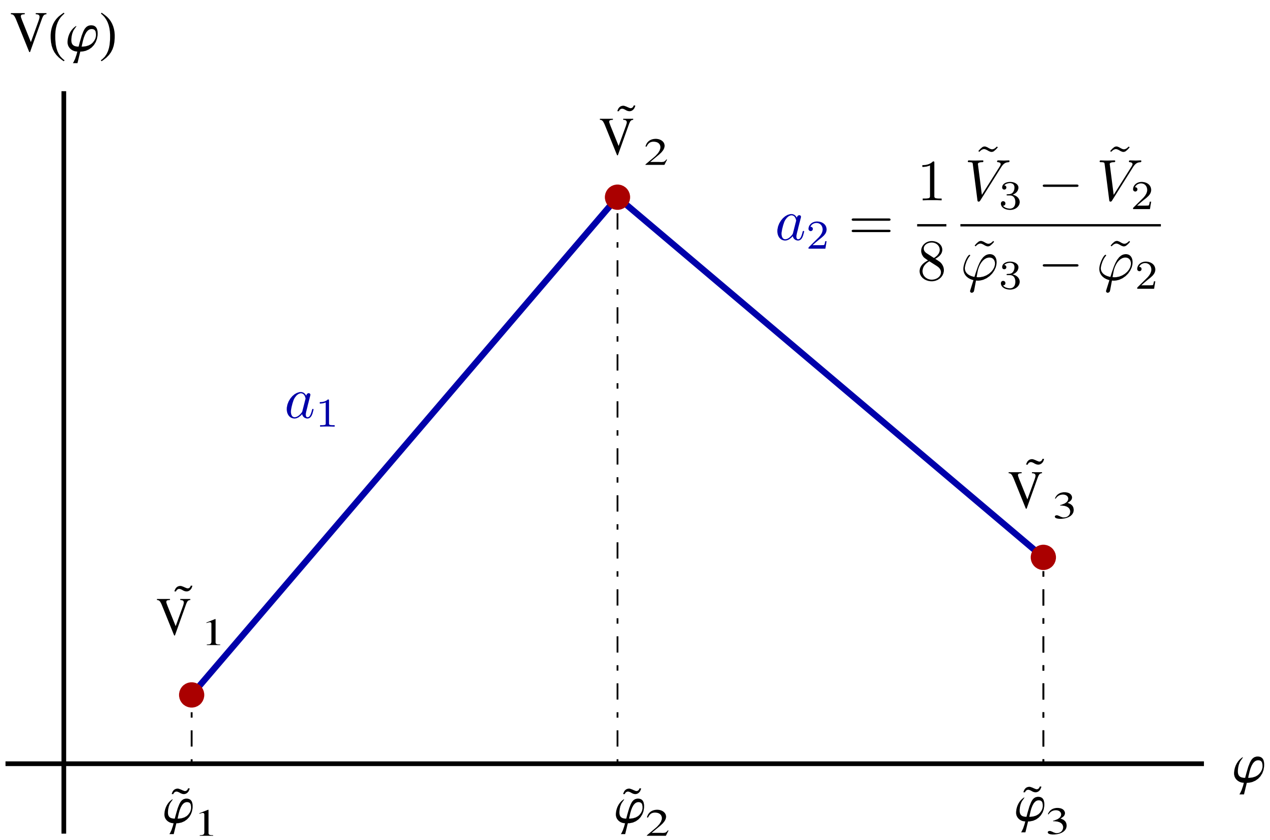
Extremize $$\ddot{\phi} + \frac{D-1}{\rho}\dot{\phi} = \frac{\mathrm{d}V}{\mathrm{d}\phi}$$

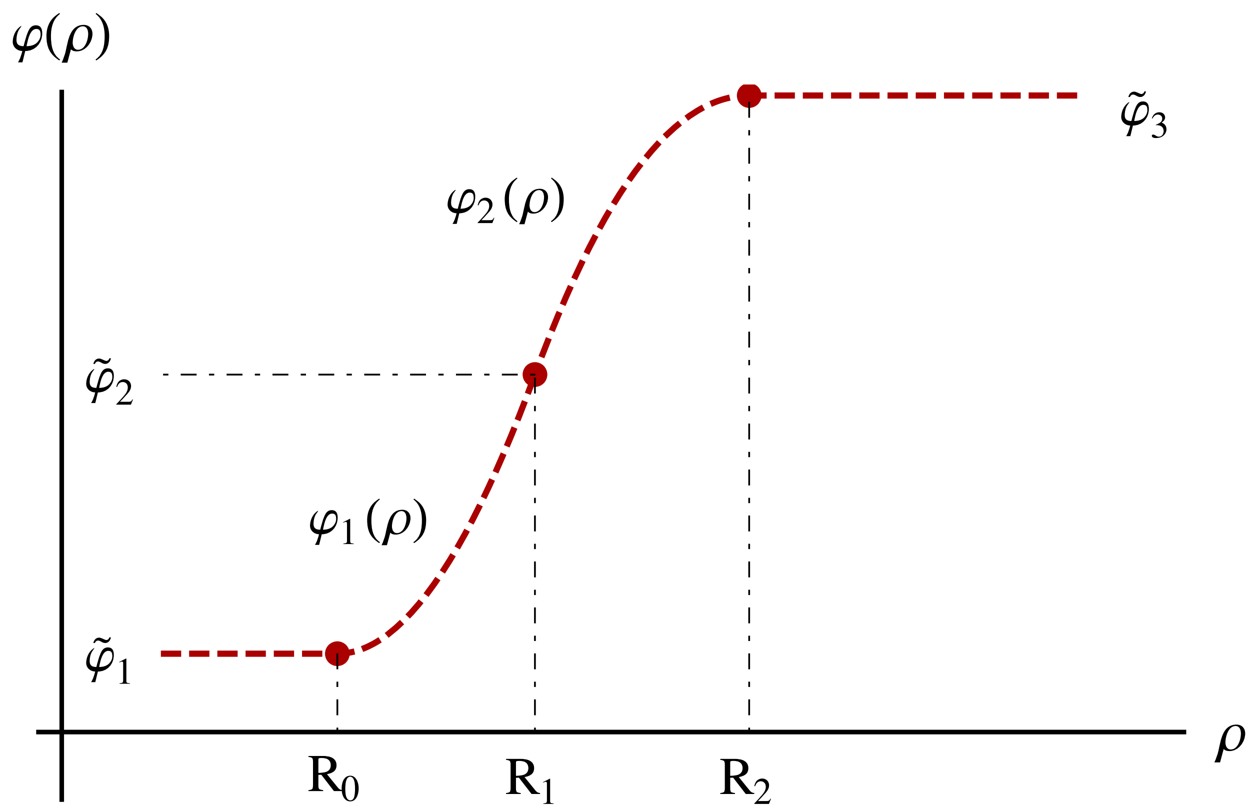
$$\phi(0) = \phi_{\rm in} , \quad \phi(\infty) = \hat{\phi}$$

$$\dot{\phi}(0) = \dot{\phi}(\infty) = 0$$



Triangular





Linear potentials

Duncan, Jensen '92

- triangle and box

Exact solution

$$\ddot\varphi + \frac{3}{\rho}\dot\varphi = dV = 8\,a$$

$$\varphi = v + a\,\rho^2 + \frac{b}{\rho^2}$$

Initial conditions @ R_0

- a) $\varphi_1(0) = \varphi_0, \quad \dot\varphi_1(0) = 0$

  $v_1 = \varphi_0, \quad b_1 = 0$

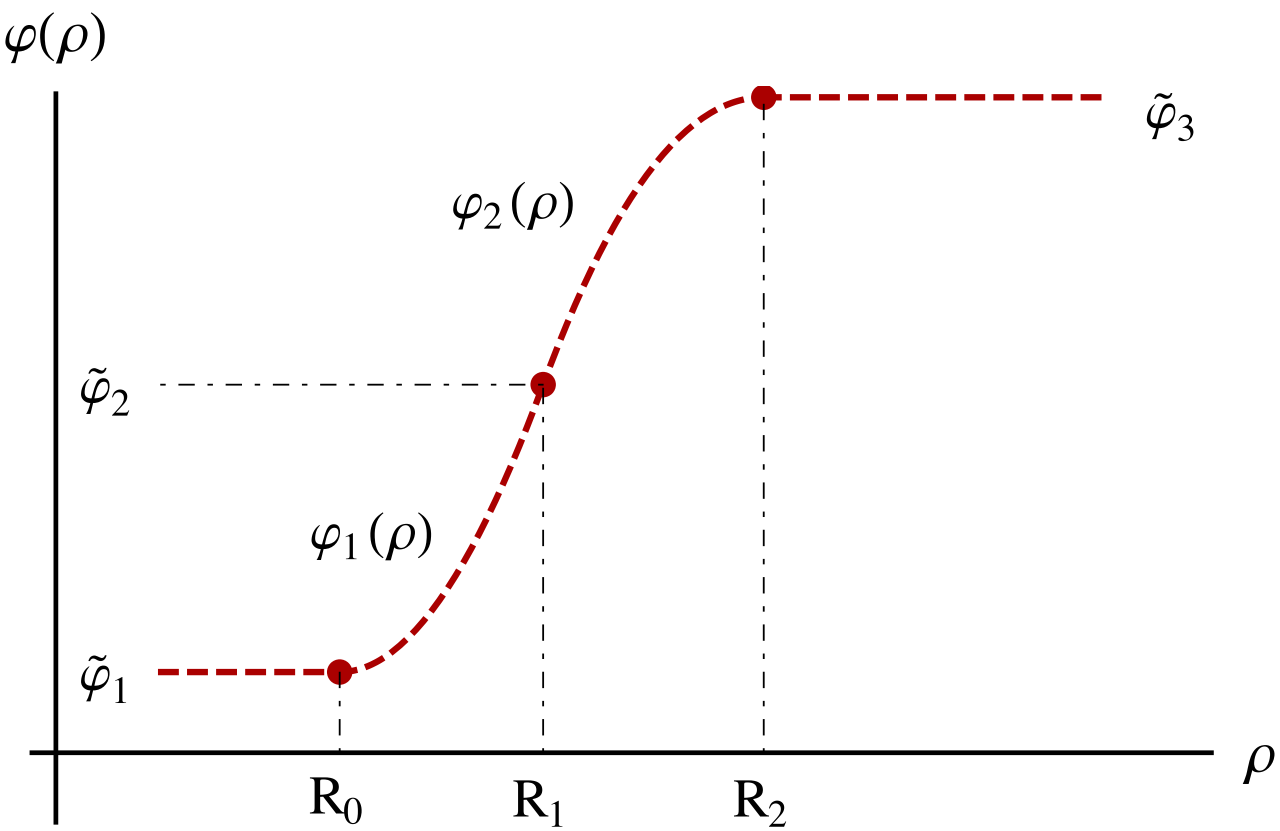
  shoot in φ_0

- b) $\varphi_1(R_0) = \tilde\varphi_1, \quad \dot\varphi_1(R_0) = 0$

  or R_0

  $v_1 = \tilde\varphi_1 - 2\,a_1\,R_0^2, \quad b_1 = a_1\,R_0^4$

Triangular



Matching conditions @ R_1

$$\varphi_1(R_1) = \varphi_2(R_1) = \tilde\varphi_2, \quad \dot\varphi_1(R_1) = \dot\varphi_2(R_1)$$

Final conditions @ R_2

$$\varphi_2(R_2) = \tilde\varphi_3, \quad \dot\varphi_2(R_2) = 0$$

$$v_2 = \tilde\varphi_3 - 2\, a_2\, R_2^2, \quad b_2 = a_2\, R_2^4$$

Complete solution - a) works in D-dimensions

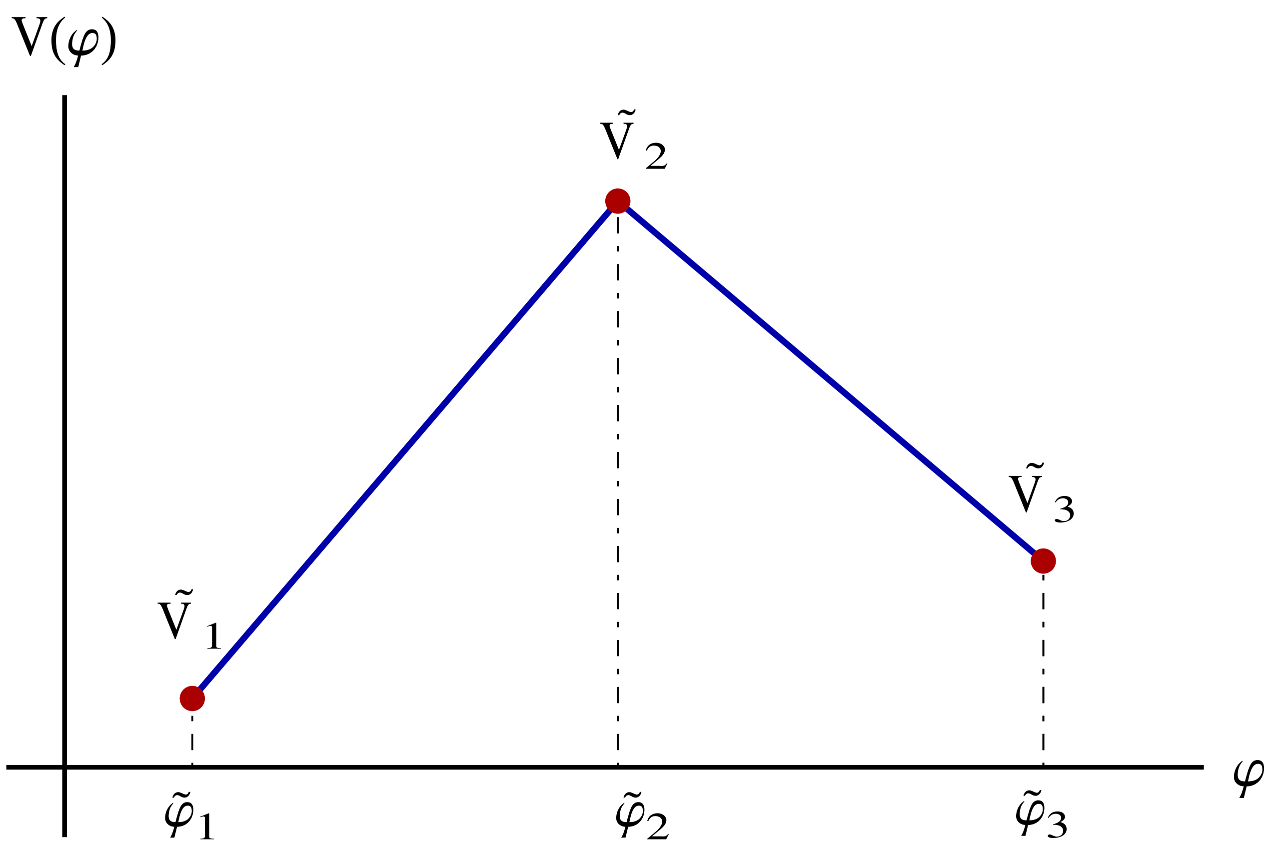
- a) $$\varphi_0 = \frac{\tilde\varphi_3 + c\,\tilde\varphi_2}{1+c}, \qquad c = 2\frac{a_2 - a_1}{a_1}\left(1 - \sqrt{\frac{a_2}{a_2 - a_1}}\right) \qquad R_1 = \sqrt{\frac{D}{4}\left(\frac{\tilde\varphi_2 - \varphi_0}{a_1}\right)}$$

- b) $$R_1 = \frac{1}{2}\frac{\tilde\varphi_3 - \tilde\varphi_1}{\sqrt{a_1\,(\tilde\varphi_2 - \tilde\varphi_1)} - \sqrt{-a_2\,(\tilde\varphi_3 - \tilde\varphi_2)}}$$

$$R_0^2 = R_1\left(R_1 - \sqrt{\frac{\tilde\varphi_2 - \tilde\varphi_1}{a_1}}\right)$$

$$R_2^2 = R_1\left(R_1 + \sqrt{\frac{\tilde\varphi_3 - \tilde\varphi_2}{-a_2}}\right)$$

Triangular summary



Complete exact analytic solution

Solved in terms of Euclidean radius

Stable in thin wall, goes over to TWA

with limited validity

Dutta, Hector, Konstandin, Vaudrevange, Westphal '12

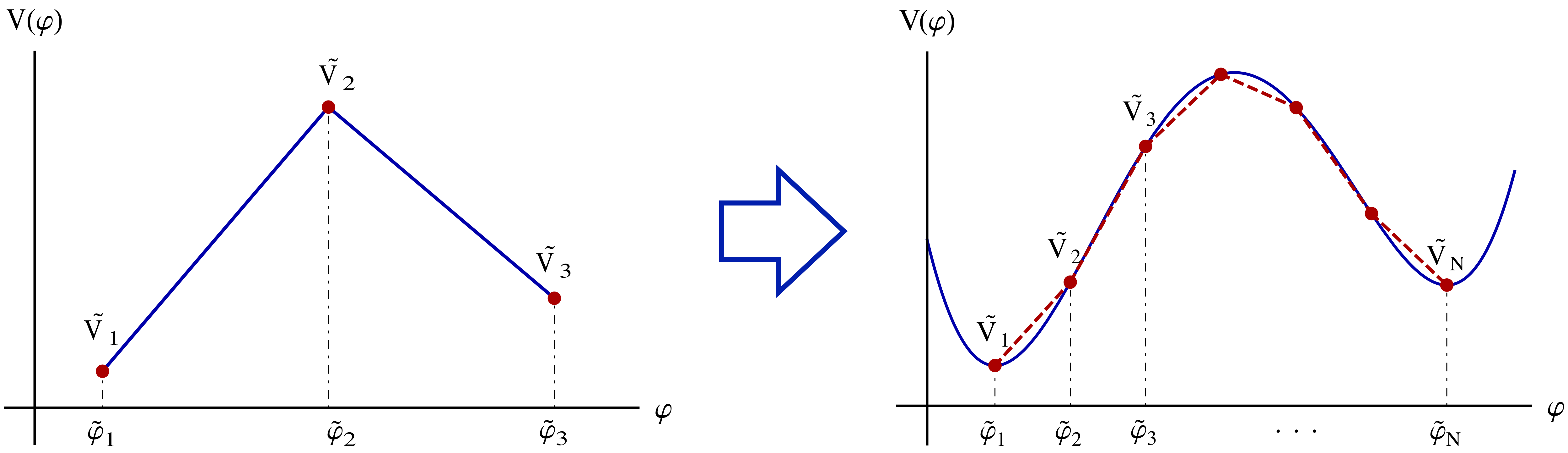
Analytic continuation in Minkowski space

describes the bubble evolution Pastras '11

Polygonal bounces

Polygonal bounces

Extend to more segments and D dimensions

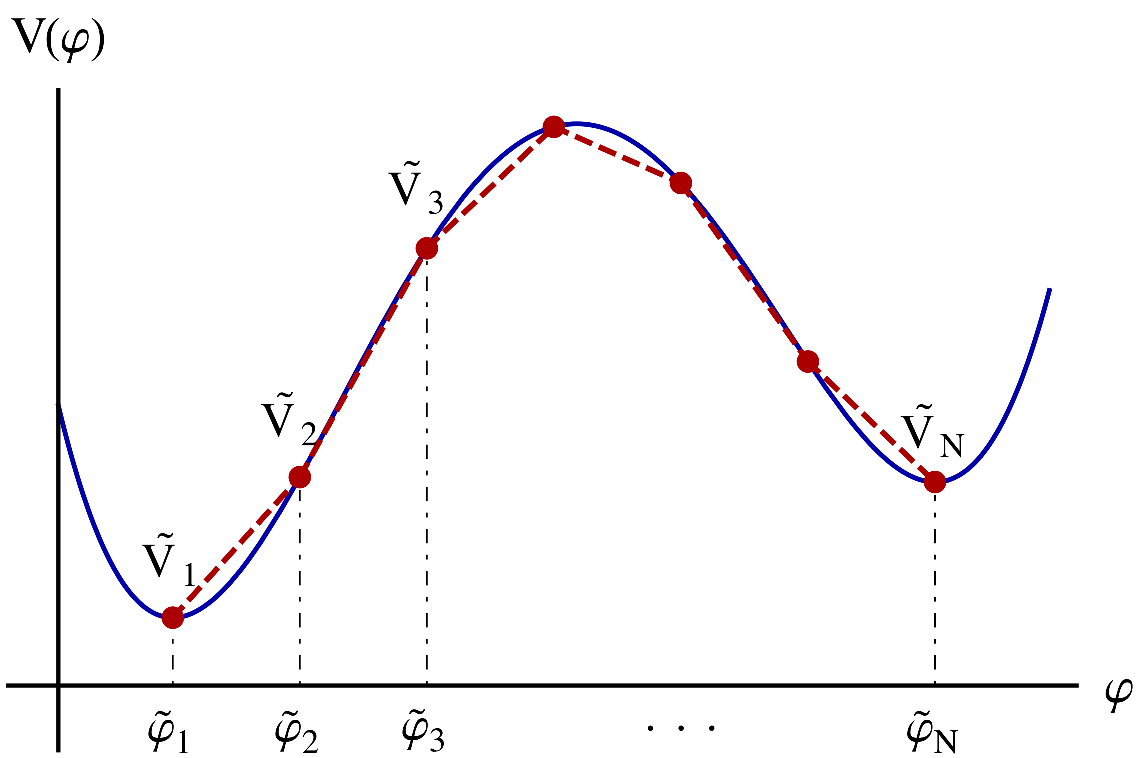


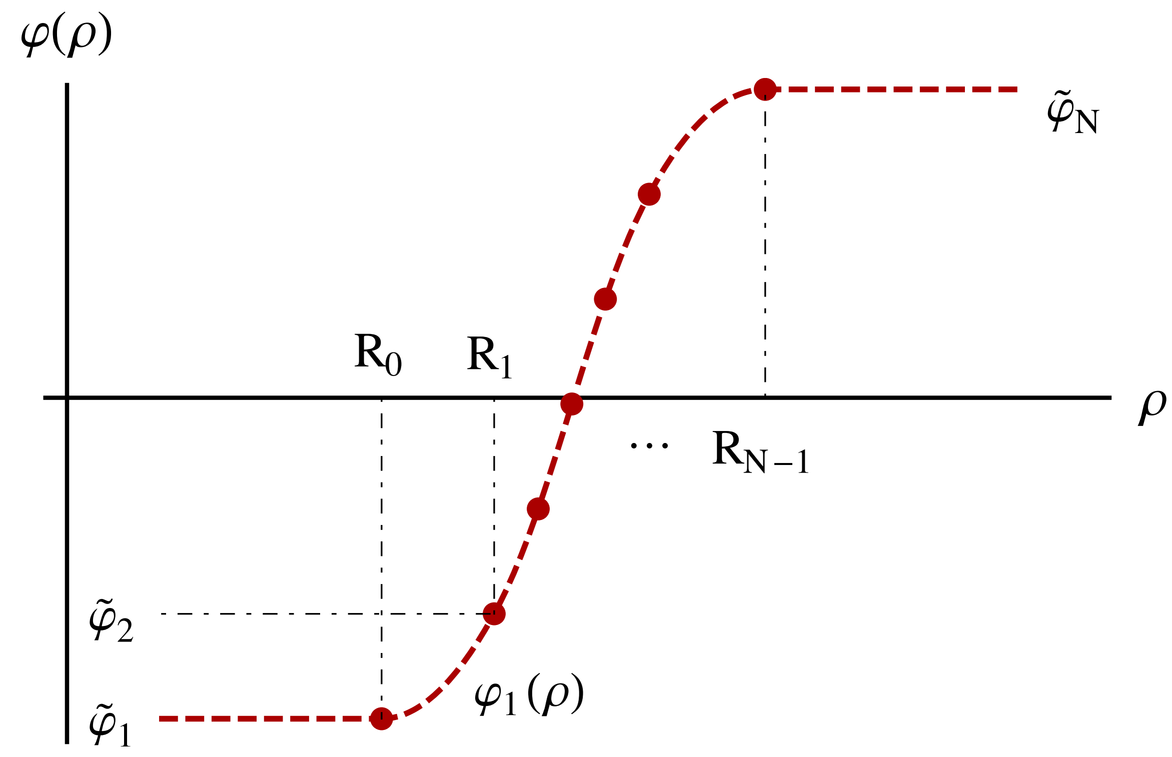
Approximates any V when $N \to \infty$, controlled precision

Geometric insight of segmentation, cover non-trivial features/unstable Vs

Semi-analytic solution for algebraic manipulation/deformation

Polygonal bounces



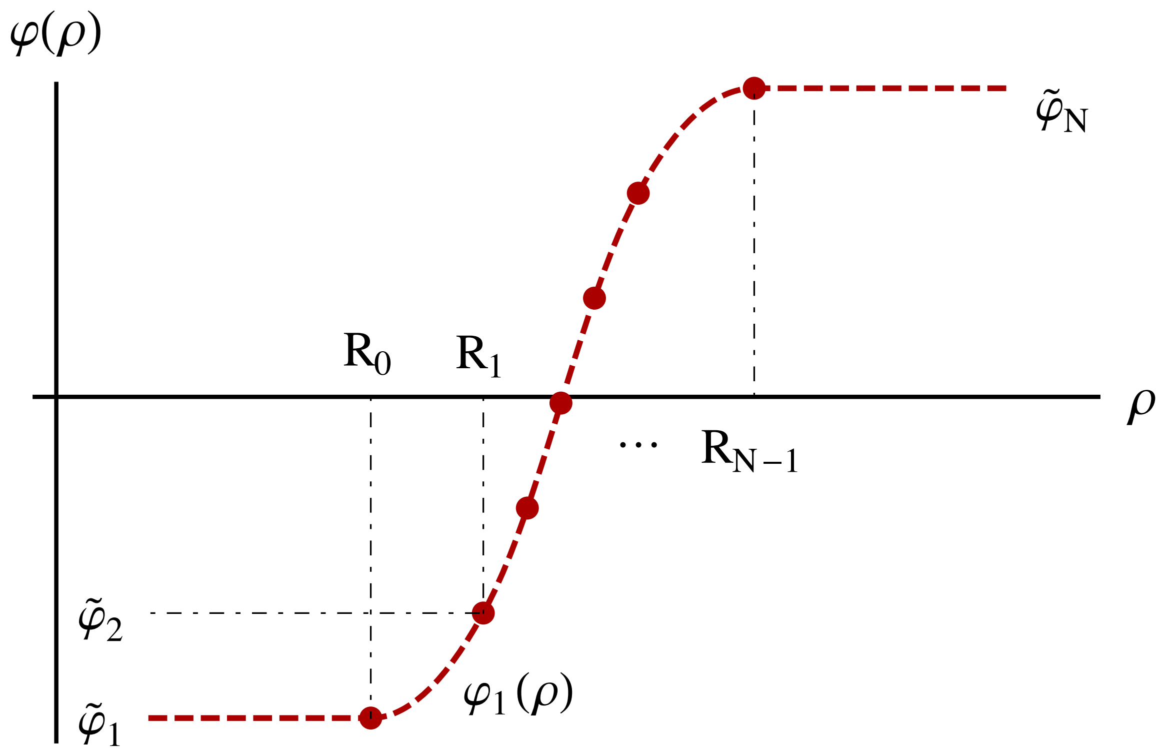


$$V_i(\varphi) = \underbrace{\left(\frac{\tilde{V}_{i+1} - \tilde{V}_i}{\tilde{\varphi}_{i+1} - \tilde{\varphi}_i}\right)}_{8\,a_i}(\varphi - \tilde{\varphi}_i) + \tilde{V}_i - \tilde{V}_N\,, \qquad dV_i = 8\,a_i.$$

No free parameters, one segment three unknowns $v_i,\ b_i,\ R_i$

Generalize case b), solve R_0 or R_i a), retrieve φ_0

Polygonal construction



$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = dV_i = 8\,a_i$$

$$\varphi_i = v_i + \frac{4}{D}a_i\,\rho^2 + \frac{2}{D-2}\frac{b_i}{\rho^{D-2}}$$

Initial/final conditions remain the same

Matching conditions @ R_i

3 parameters and 3 unknowns/segment

$$\varphi_i(R_1) = \varphi_{i+1}(R_i) = \tilde{\varphi}_{i+1}, \quad \dot{\varphi}_i(R_i) = \dot{\varphi}_{i+1}(R_i)$$

The bounce defined recursively

- a) $R_0 = 0$

$$v_n = \varphi_0 - \frac{4}{D-2}\left(a_1 R_0^2 + \sum_{i=1}^{n-1}(a_{i+1} - a_i)\,R_i^2\right)$$

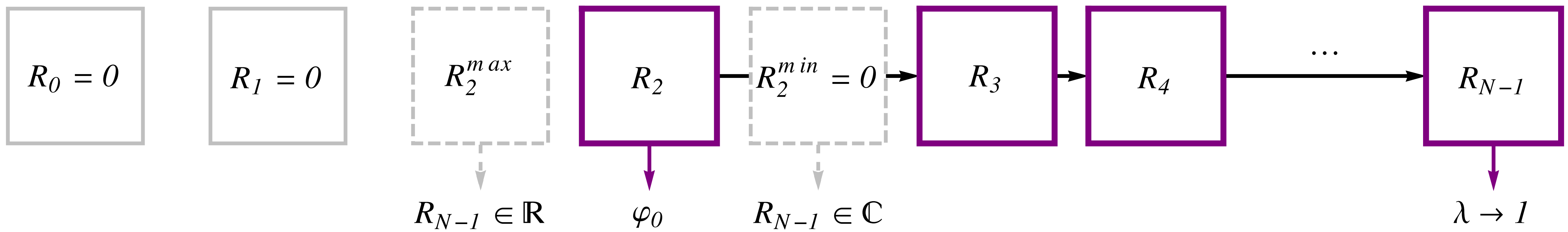
- b) $\varphi_0 = \tilde{\varphi}_1$

$$b_n = \frac{4}{D}\left(a_1 R_0^D + \sum_{i=1}^{n-1}(a_{i+1} - a_i)\,R_i^D\right)$$

Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

Fewnomials $$R_n^D - \frac{D}{4}\frac{\delta_n}{a_n} R_n^{D-2} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0$$ $\delta_n = \tilde{\varphi}_{n+1} - v_n$

require real positive roots

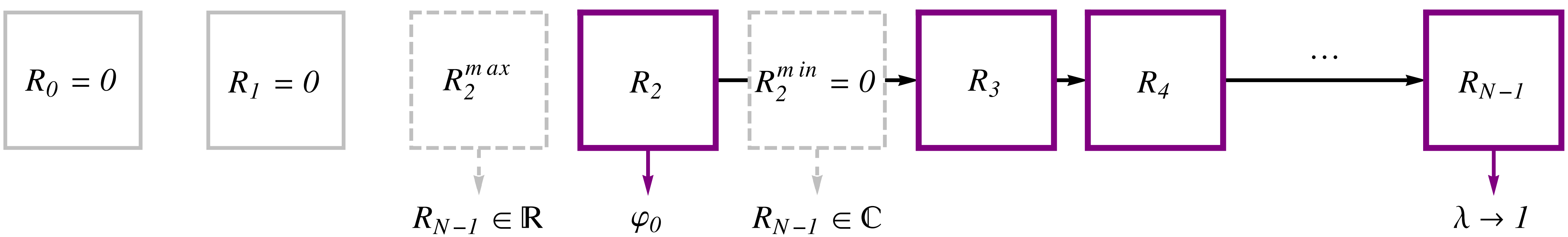


Radii computed at each segment from matching the fields $\varphi_n(R_n) = \tilde{\varphi}_{n+1}$

Fewnomials

$$R_n^D - \frac{D}{4}\frac{\delta_n}{a_n}R_n^{D-2} + \frac{D}{2(D-2)}\frac{b_n}{a_n} = 0 \qquad \delta_n = \tilde{\varphi}_{n+1} - v_n$$

require real positive roots



radii solutions

$$D = 3: \quad 2R_n = \frac{1}{\sqrt{a_n}}\left(\frac{\delta_n}{\xi} + \xi\right), \qquad \xi^3 = \sqrt{36 a_n b_n^2 - \delta_n^3} - 6\sqrt{a_n} b_n,$$

simple cubic

$$D = 4: \quad 2R_n^2 = \frac{1}{a_n}\left(\delta_n + \sqrt{\delta_n^2 - 4 a_n b_n}\right)$$ quadratic

$D = 2, 6, 8$ in the paper, other Ds possible numerically Amariti '20

Bounce action

Euclidean action

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1} \, \mathrm{d}\rho \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

PB action

$$S_{>2} = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \left\{ \frac{R_0^D}{D}\left(\tilde{V}_1 - \tilde{V}_N\right) + \sum_{i=1}^{N-1} \left[\rho^2 \left(\frac{32 a_i^2 (D+1)\rho^D}{D^2(D+2)} + \frac{16 a_i b_i}{D(D-2)} - \frac{2b_i^2}{\rho^D(D-2)} \right) + \frac{\rho^D}{D}\left(8 a_i \left(v_i - \tilde{\varphi}_i\right) + \tilde{V}_i - \tilde{V}_N\right) \right]_{R_{i-1}}^{R_i} \right\}$$

Total

$$S = \mathcal{T} + \mathcal{V}$$

$$\mathcal{T} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho \, \dot{\varphi}^2, \qquad \mathcal{V} \propto \int_0^\infty \rho^{D-1} \mathrm{d}\rho \, \mathrm{V}(\varphi)$$

kinetic potential

Derrick's theorem

Non-existence of non-trivial static solutions of KG equation, no solitonic scalar 'particles'

Derrick '64

Unstable under re-scaling $\varphi(\rho) \to \varphi(\rho/\lambda)$

$$\lambda \times 0 = 0,$$
$$\lambda \times \infty = \infty$$

change of variables,
int. limits remain the same

$$S_D^{(\lambda)} = \lambda^{D-2}\mathcal{T} + \lambda^D \mathcal{V}$$

action is extremized
at non-scaled values
for true solutions

$$\left.\frac{dS_D^{(\lambda)}}{d\lambda}\right|_{\lambda=1} = 0$$

$$(D-2)\mathcal{T} + D\mathcal{V} = 0$$

relation between kinetic and potential

$$\left.\frac{d^2 S_D^{(\lambda)}}{d\lambda^2}\right|_{\lambda=1} < 0$$

Caveat for PB $R \to \lambda R$

Works for $N \gg 1$

Benchmarks

Linearly off-set quartic potential

Coleman '77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right)$$

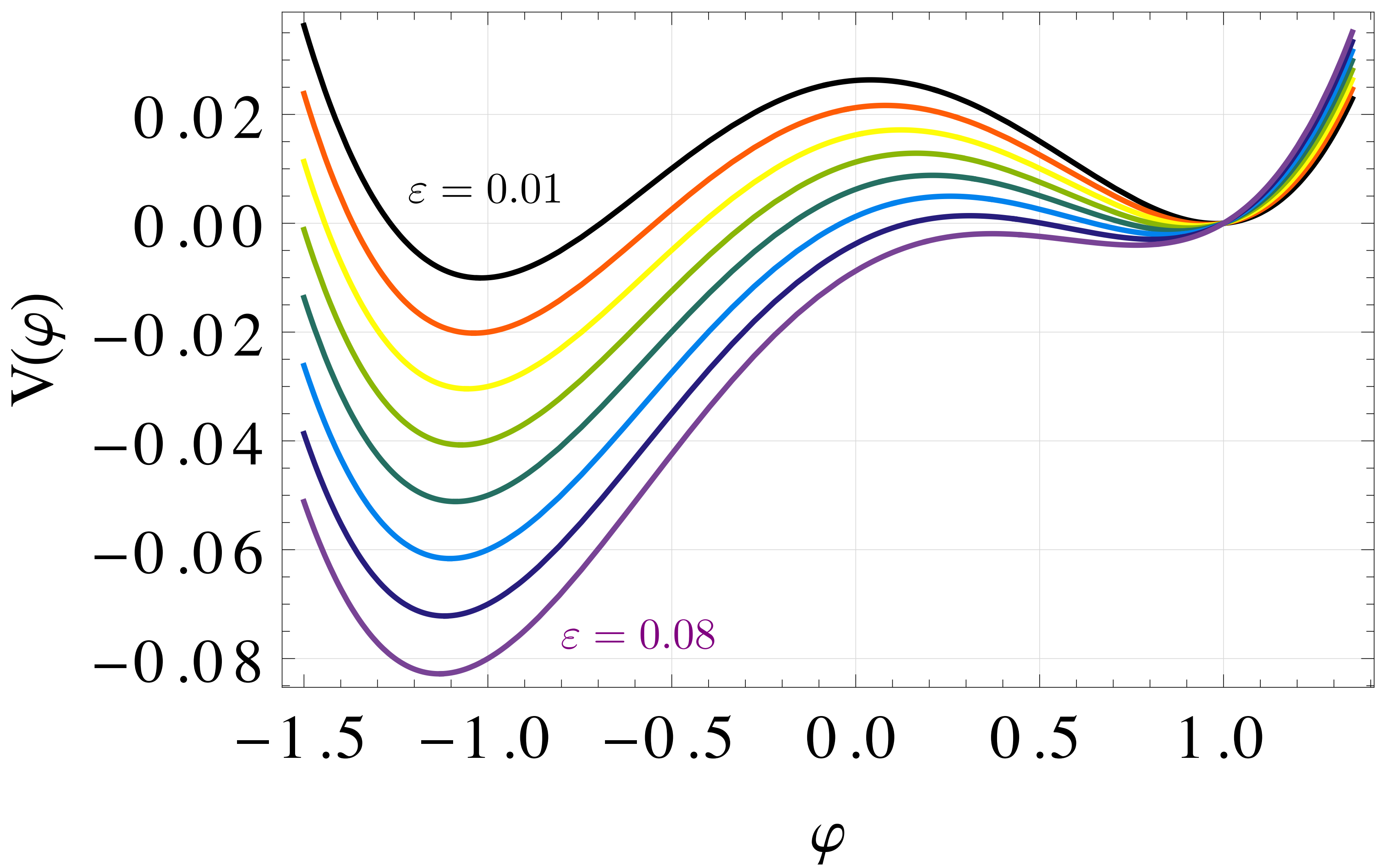
Benchmark for testing $\lambda = 0.25, \quad v = 1$

rescaling

Sarid '98

$\varepsilon \to 0$

thin wall



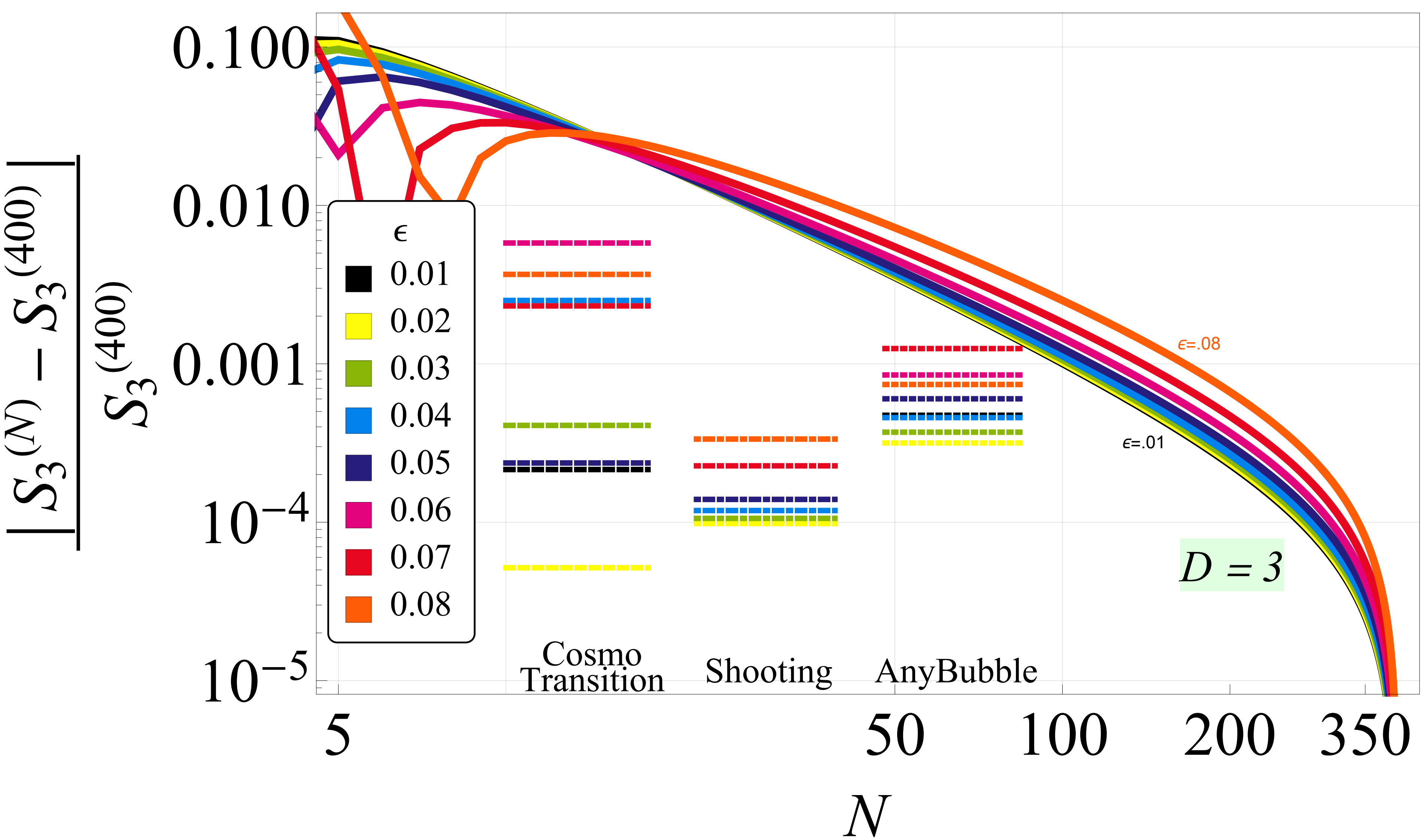
thick wall

$\varepsilon > 0.8$

2nd minimum disappears

Euclidean action, comparisons

- CosmoTransitions — Runge-Kutta PDE solver, initial value approximations — Wainwright '11
- AnyBubble — multiple shooting, damping approximations — Masoumi, Olum, Shlaer '16
- Shooting — Mathematica, precise setting of initial values, issues with 0, infinity



PB within permille after 100 iterations

stable for thin/ thick

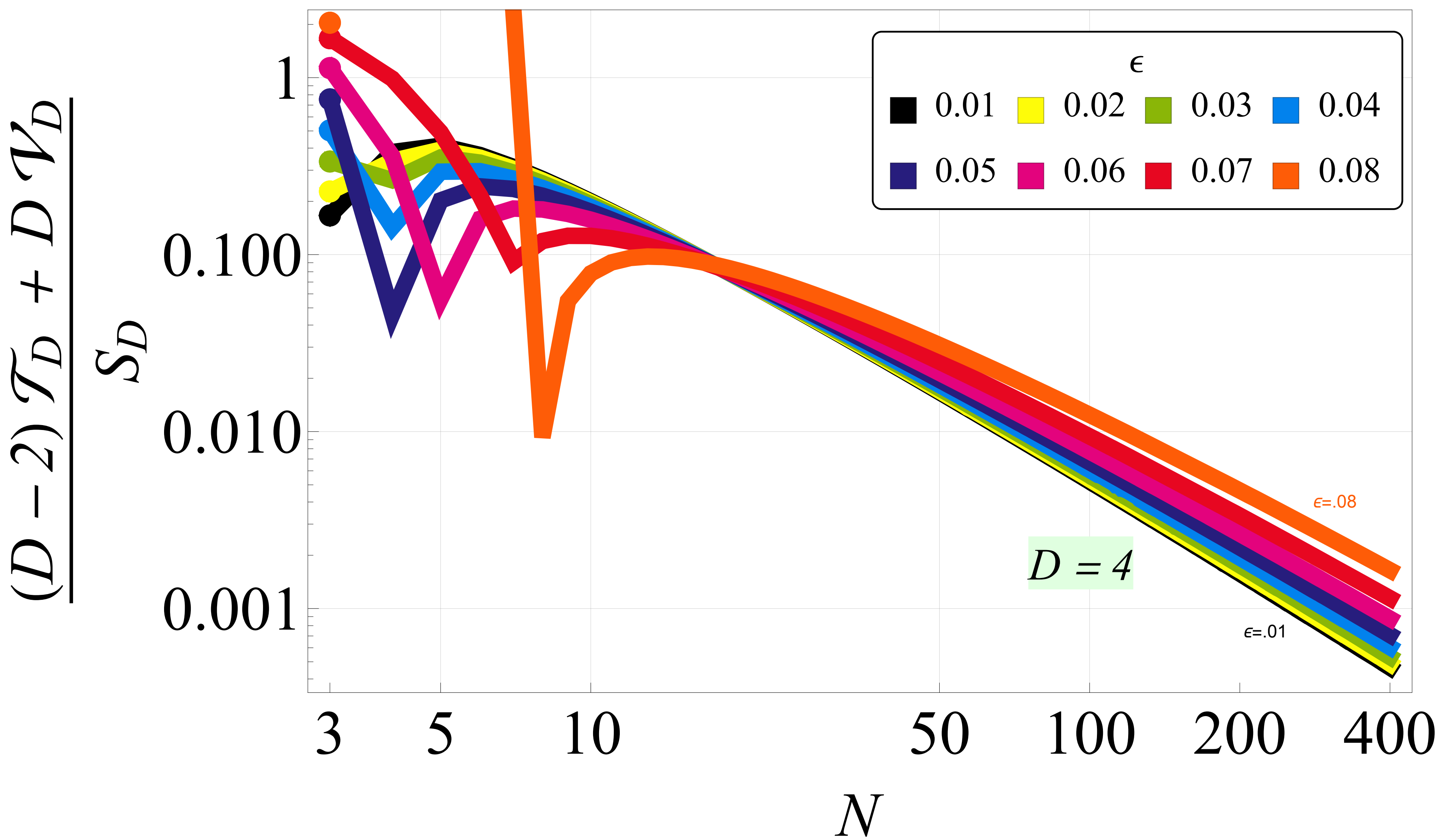
faster convergence for thin wall

Derrick's theorem $(D-2)\mathcal{T} + D\mathcal{V} \to 0$

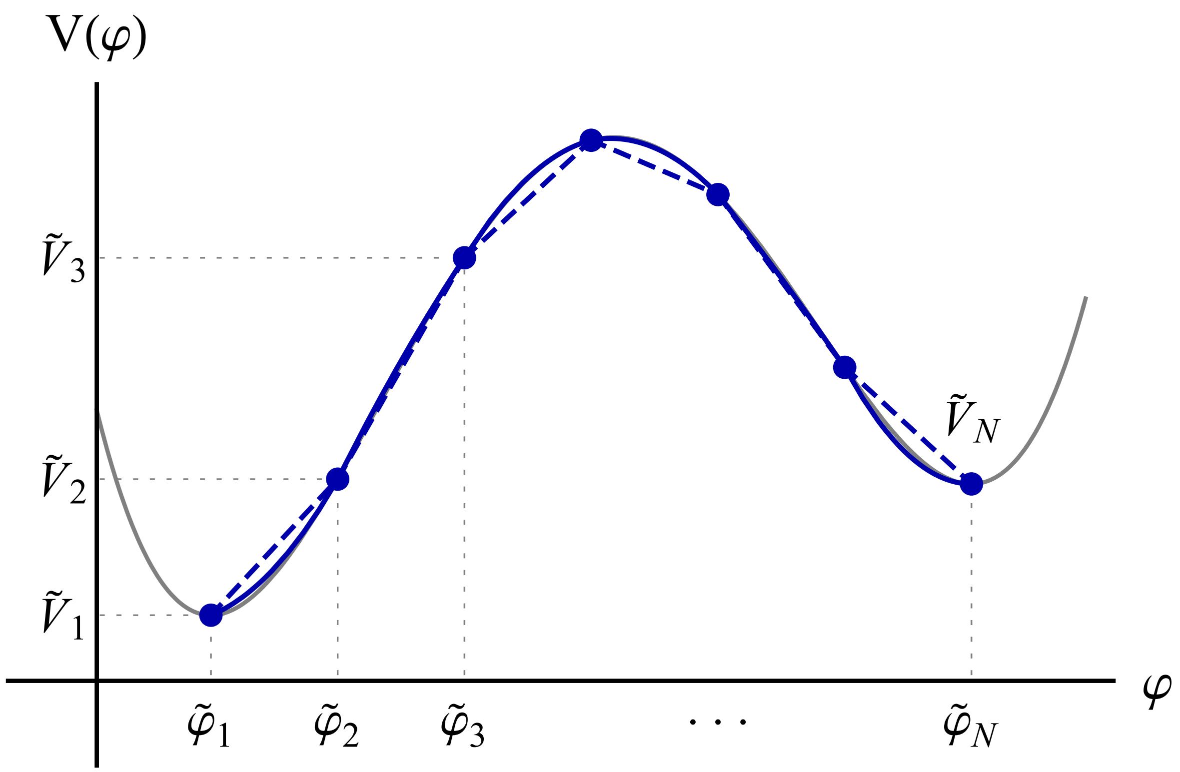
finite part corrections up to $N \simeq 10$

independent measure of goodness of approximation

above relation 'exact' for the PB potential



Higher orders



Expand to higher orders

- improves convergence
- important @ extrema

$$V_i \simeq \tilde{V}_i - \tilde{V}_N + \partial\tilde{V}_i\left(\varphi_i - \tilde{\varphi}_i\right)$$

$$+ \frac{\partial^2\tilde{V}_i}{2}\left(\varphi_i - \tilde{\varphi}_i\right)^2$$

Perturbative expansion $\varphi = \varphi_{PB} + \xi$

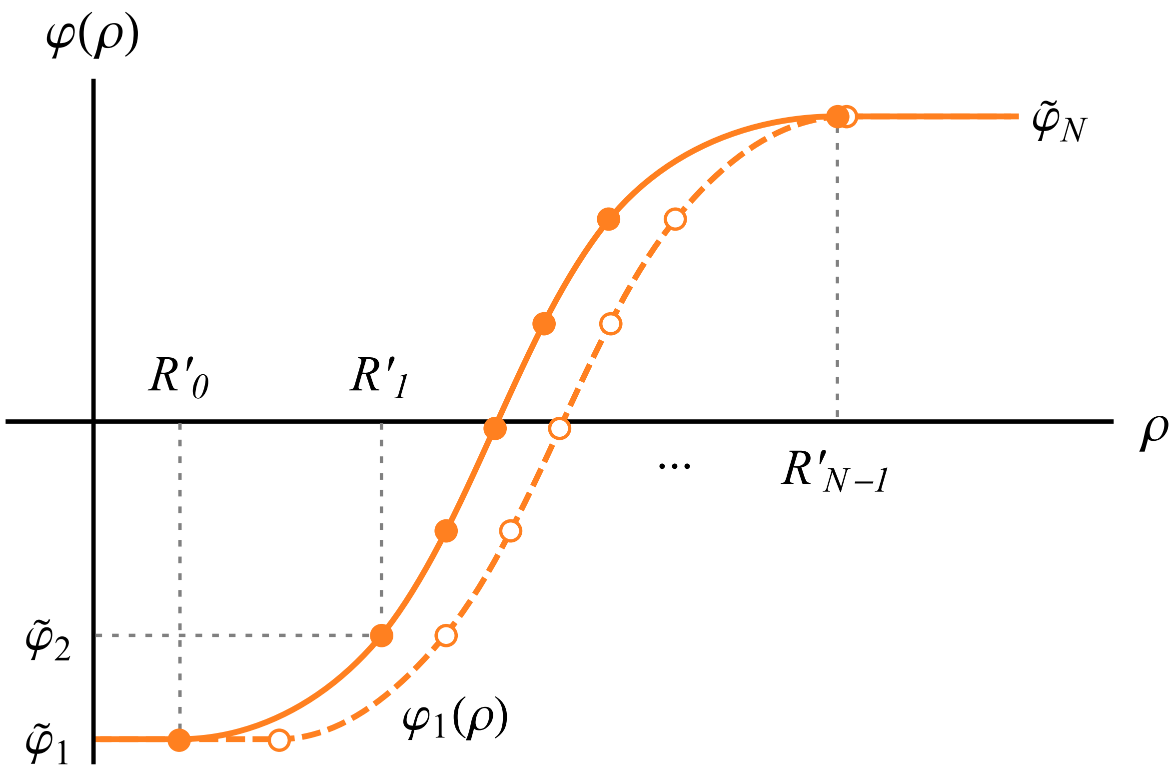
$$\ddot{\varphi} + \frac{D-1}{\rho}\dot{\varphi} = 8\left(a + \alpha\right) + \delta dV\left(\varphi_{PB}(\rho)\right)$$

$$\ddot{\xi} + \frac{D-1}{\rho}\dot{\xi} = 8\alpha + \delta dV(\rho)$$

$$\mathcal{I}_s^{D=4} = \partial^2\tilde{V}_s\left(\frac{v_s - \tilde{\varphi}_s}{8}\rho^2 + \frac{b_s}{2}\ln\rho + \frac{a_s}{24}\rho^4\right)$$

$$\xi = \nu + \frac{2}{D-2}\frac{\beta}{\rho^{D-2}} + \frac{4}{D}\alpha\rho^2 + \mathcal{I}(\rho)$$

Higher orders



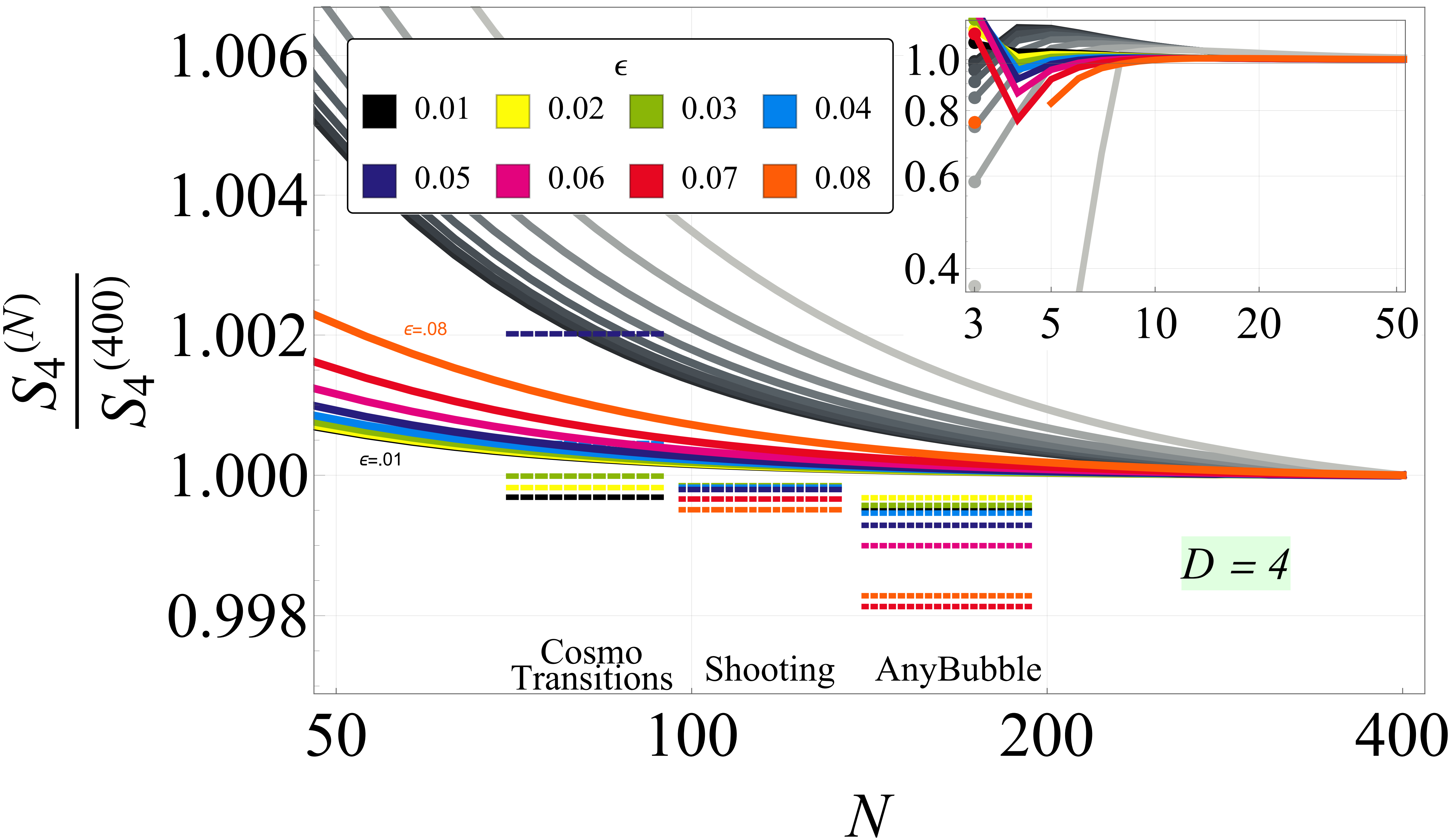
Match at perturbed radii

$$R_s \to R_s\,(1 + r_s)\,, \qquad r_s \ll 1$$

Rederive the matching conditions

A single linear equation = very fast

$$r_s = \frac{\beta_s + \frac{D-2}{2}\left(\nu_s + \mathcal{I}_s + \frac{4}{D}\alpha_s R_s^2\right) R_s^{D-2}}{(D-2)\left(b_s - \frac{4}{D} a_s R_s^D\right)}$$



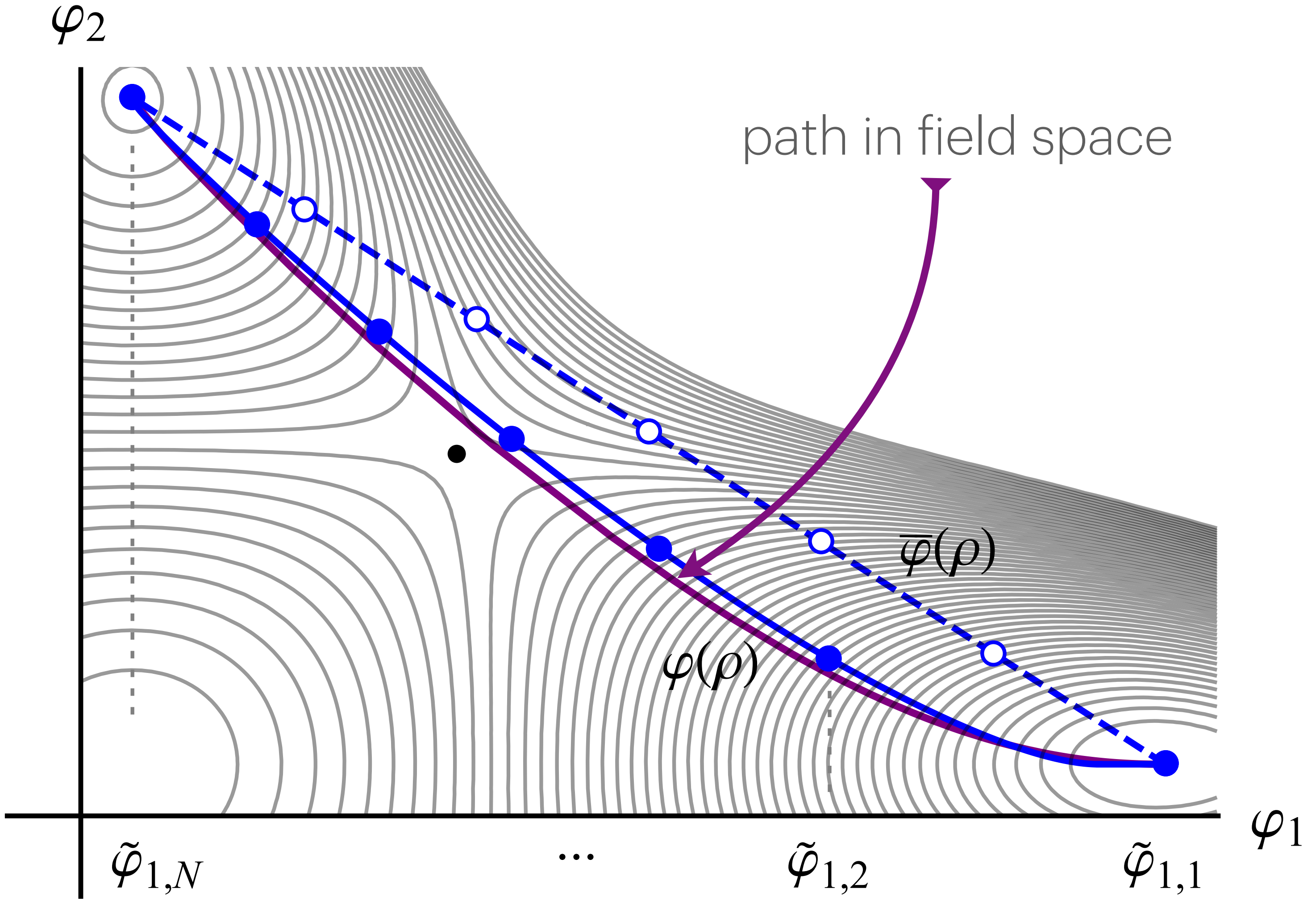
ϵ
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
ϵ=.08
ϵ=.01
Cosmo Transitions
Shooting
AnyBubble
D = 4
N
50
100
200
400
1.006
1.004
1.002
1.000
0.998
1.0
0.8
0.6
0.4
3
5
10
20
50

Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{dV}{d\varphi_i} \qquad \varphi_i(0) = \varphi_{0i}$$

Shooting method impractical

- highly non-linear
- multi-dimensional field space



Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{dV}{d\varphi_i} \qquad \varphi_i(0) = \varphi_{0i}$$

- CosmoTransitions — Wainwright '11

  bounce and path deformation separate, oscillations, Runge-Kutta PDE solver

- AnyBubble — Masoumi, Olum, Shlaer '16

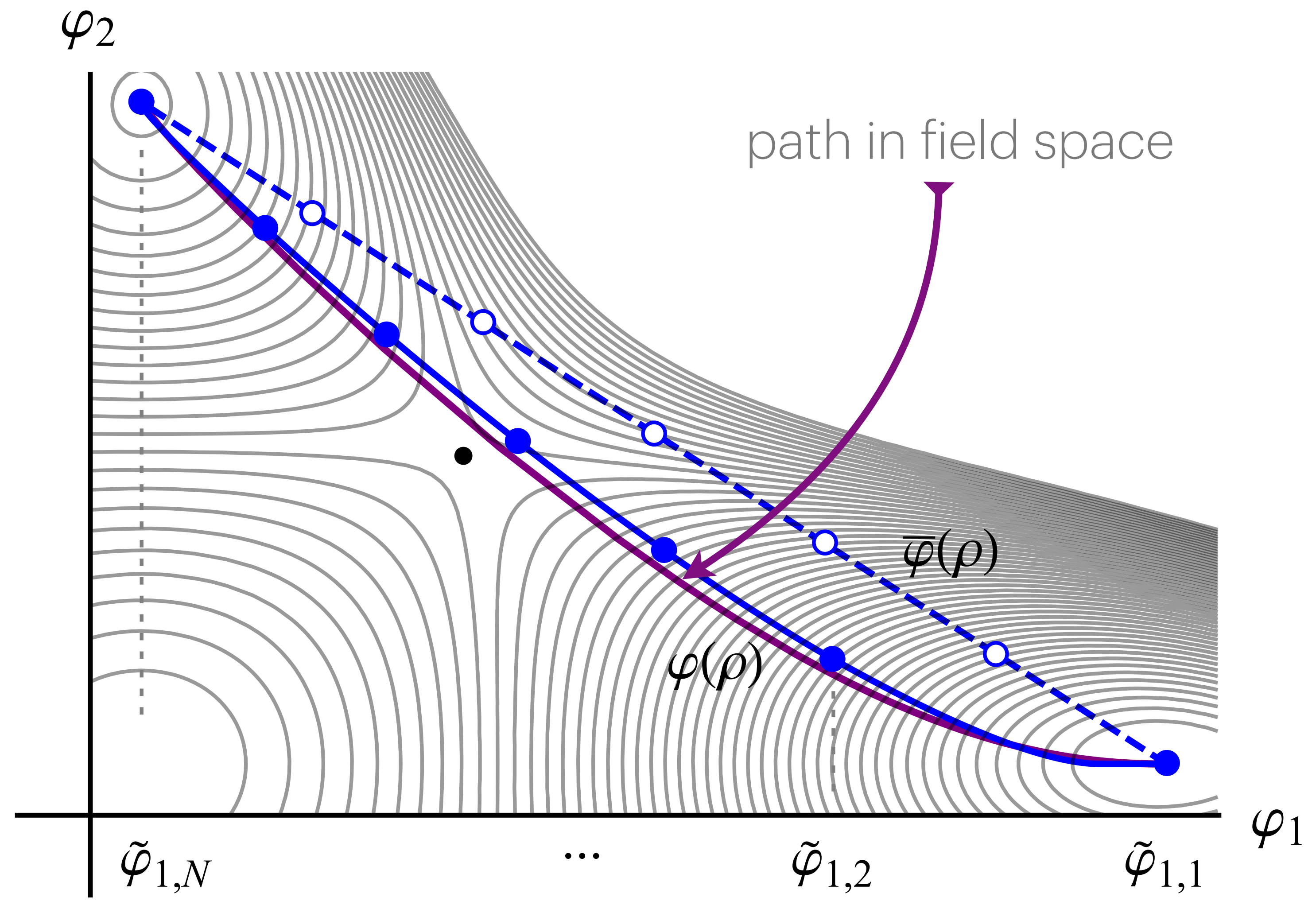
  multiple shooting, damping approximations

- Other approaches

  tunnelling potential — Espinosa, Konstandin '18

  machine learning — Piscopo, Spannowsky, Waite '19

  gradient flow — Sato '19

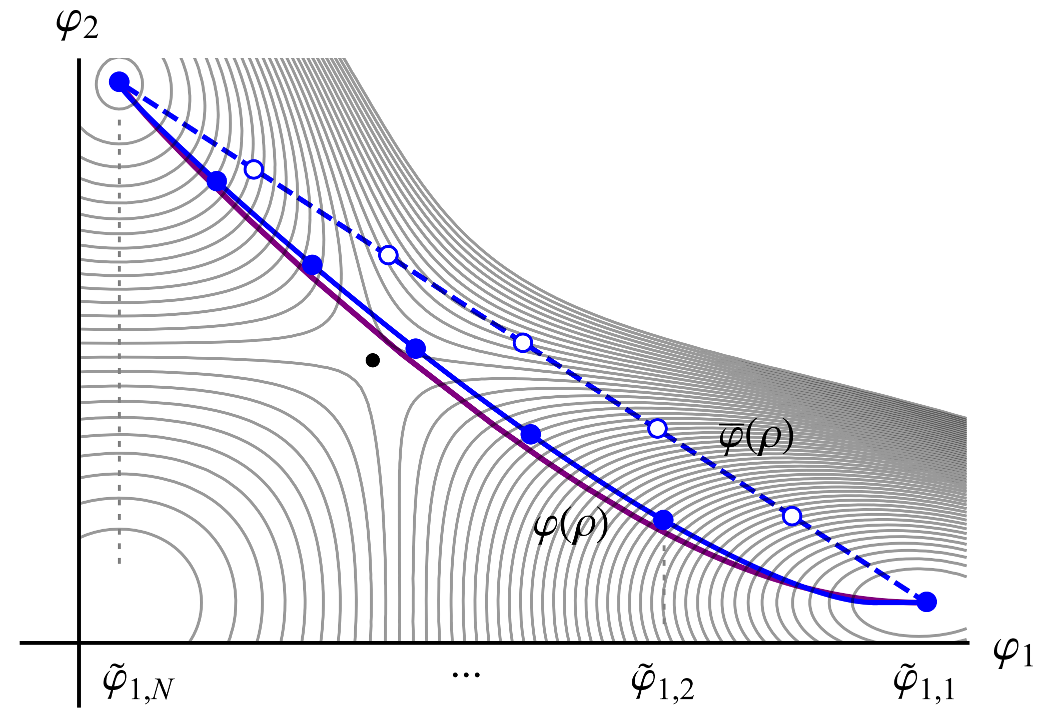


Polygonal approach with many fields

- Initial ansatz — straight line, via saddle, custom segmentation
- Initial solution — longitudinal single field PB

$\tilde{\varphi}_{is}$

$\bar{\varphi}_{is}(\rho)$



Crucial idea #1

- perturbation up to linear term in V, keeps the PB

$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i}\left(\bar{\varphi} + \zeta\right)$$

$$\zeta_{is} = v_{is} + \frac{2}{D-2}\frac{b_{is}}{\rho^{D-2}} + \frac{4}{D}a_{is}\rho^2$$

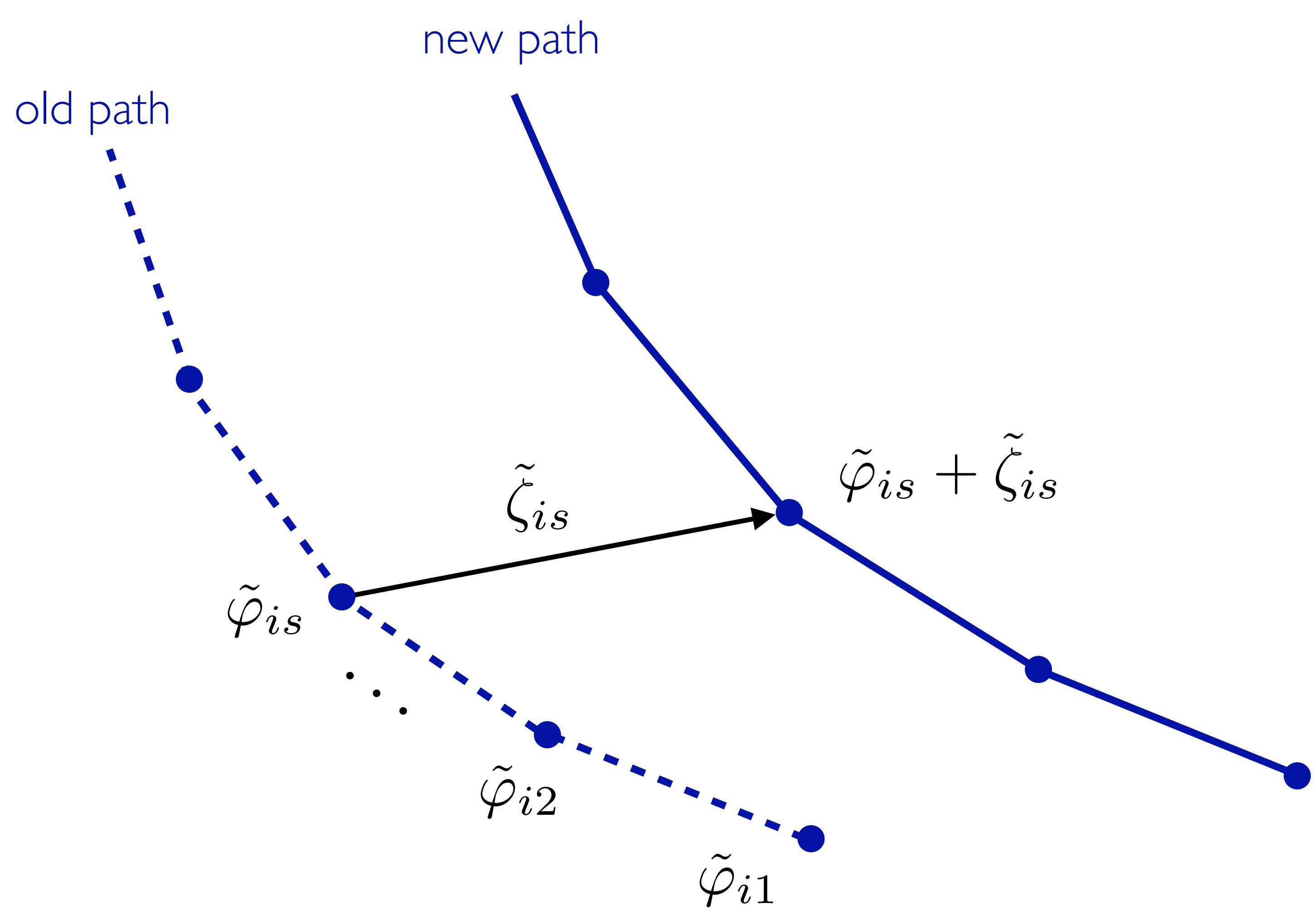
$$\underbrace{\ddot{\bar{\varphi}}_{is} + \frac{D-1}{\rho}\dot{\bar{\varphi}}_{is}}_{8\bar{a}_{is}} + \underbrace{\ddot{\zeta}_{is} + \frac{D-1}{\rho}\dot{\zeta}_{is}}_{8a_{is}} = \frac{dV}{d\varphi_i}(\bar{\varphi} + \zeta)$$

Crucial idea #2

$$8a_{is} \simeq \frac{dV}{d\varphi_i}\left(\tilde{\varphi}_{is} + \tilde{\zeta}_{is}\right) - 8\bar{a}_{is}$$

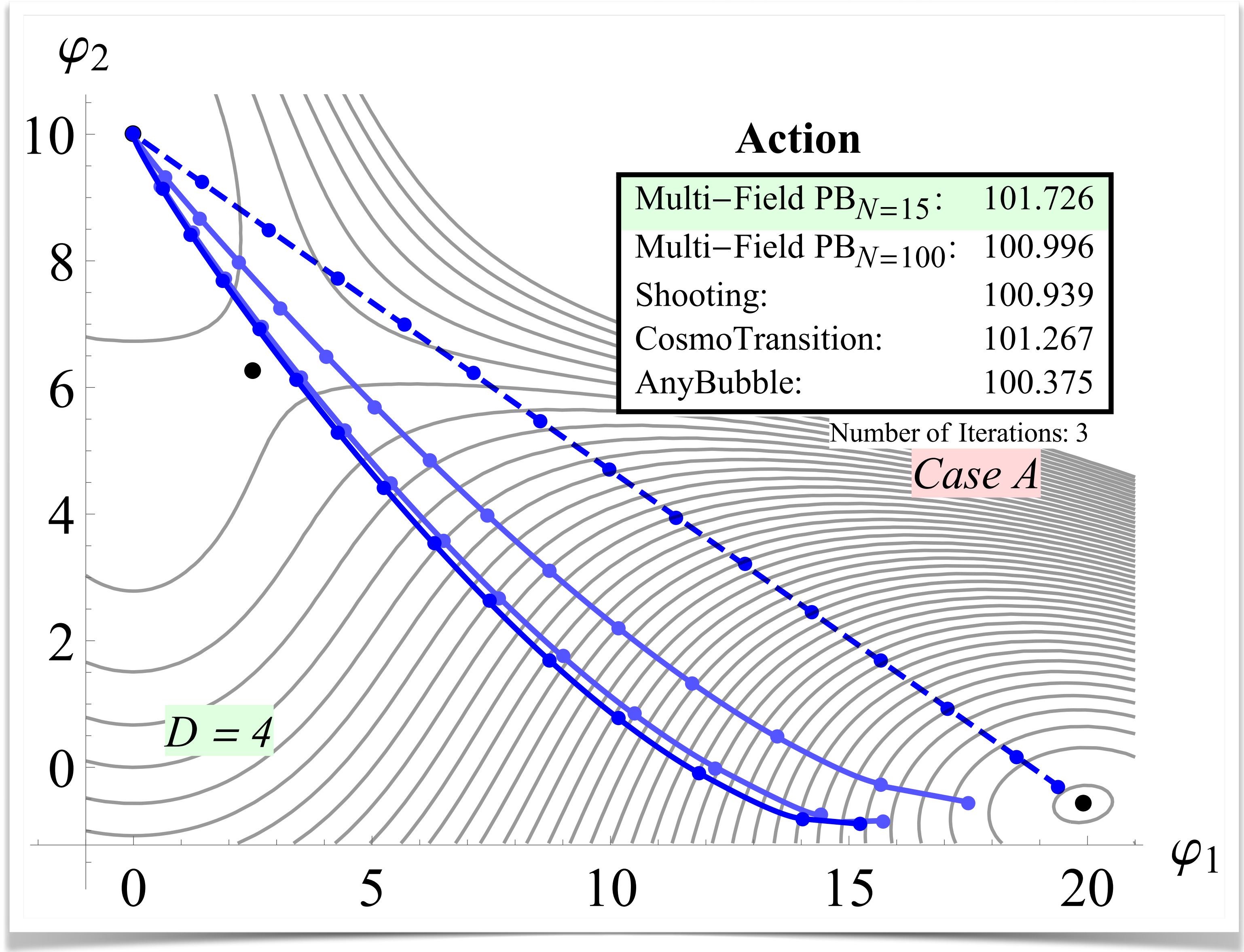
$$\frac{dV}{d\varphi_i} \simeq \frac{1}{2}\left(d_i\tilde{V}_s + d_i\tilde{V}_{s+1} + d^2_{ij}\tilde{V}_s\tilde{\zeta}_{js} + d^2_{ij}\tilde{V}_{s+1}\tilde{\zeta}_{js+1}\right)$$

- simultaneous solution for the bounce and path deformation
- linear system for r_{i0} (as in the single field expansion) and $\tilde{\zeta}_{is}$
- iterate until $\tilde{\zeta}_{is} < \varepsilon_{\Delta\varphi}$



$$V(\varphi_i) = \sum_{i=1}^{2} \left(-\mu_i^2 \varphi_i^2 + \lambda_i^2 \varphi_i^4\right) + \lambda_{12} \varphi_1^2 \varphi_2^2 + \tilde{\mu}^3 \varphi_2$$

- no oscillations
- converges in a few iterations
- works for thin wall
- works for *D*=3 and *4*
- tested for up to 20 fields



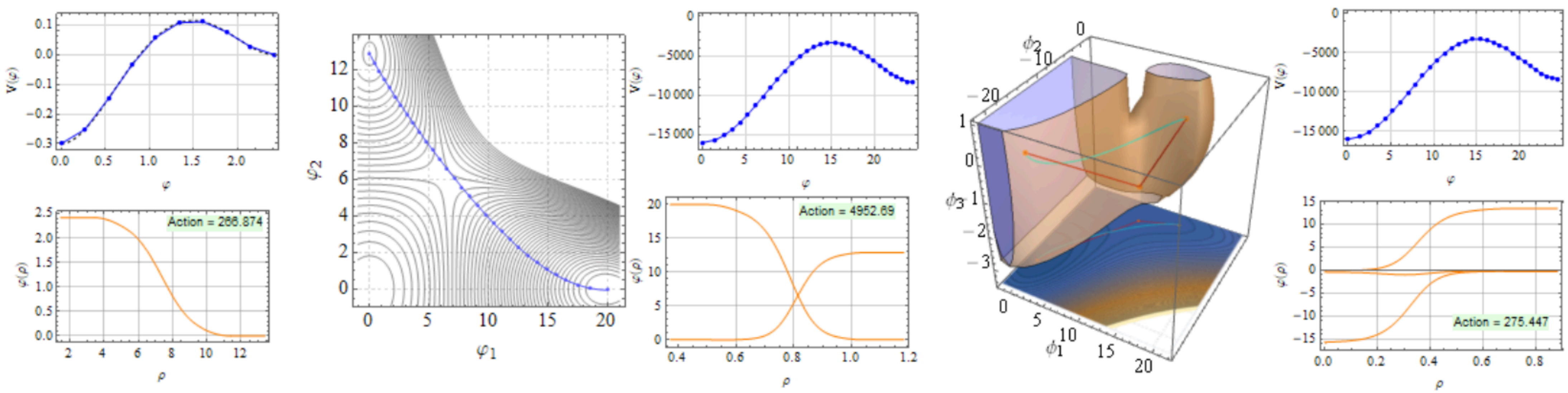
FindBounce

Guada, MN, Pintar '20

FindBounce https://github.com/vguada/FindBounce/releases

FindBounce is a Mathematica package that computes the bounce configuration needed to compute the false vacuum decay rate with multiple scalar fields.

We kindly ask the users of the package to cite the two papers that describe the working of the *FindBounce* package: the paper with the original proposal by Guada, Maiezza and Nemevšek (2019) and the software release manual by Guada, Nemevšek and Pintar (2020).



Installation

To use the *FindBounce* package you need Mathematica version 10.0 or later. The package is released in the `.paclet` file format that contains the code, documentation and other necessary resources. Download the latest `.paclet` file from the repository "releases" page to your computer and install it by evaluating the following command in the Mathematica:

```
(* Path to .paclet file downloaded from repository "releases" page. *)
PacletInstall["full/path/to/FindBounce-X.Y.Z.paclet"]
```

Load the package as usual

```
In[1]:= Needs["FindBounce`"]
```

Define a metastable potential

```
In[2]:= V[x_]:= 0.5 x^2 + 0.5 x^3 + 0.12 x^4;

In[3]:= extrema = x/.Sort@Solve[D[V[x],x]==0];
```

Compute the bounce - obtain bf = the bounce function

```
In[4]:= bf = FindBounce[V[x],x,{extrema[[1]],extrema[[3]]}]
```

Out[4]= BounceFunction[ Action: 73 500. Dimension: 4]

```
In[5]:= bf["Action"]

Out[5]= 73496.

In[6]:= bf["Dimension"]

Out[6]= 4
```

Retrieve the bounce properties

Run on single points `FindBounce[{{x1,V1},{x2,V2},...}]`

And multifields `FindBounce[V[x,y,...],{x,y,...},{m1,m2}]`

Custom options control the input

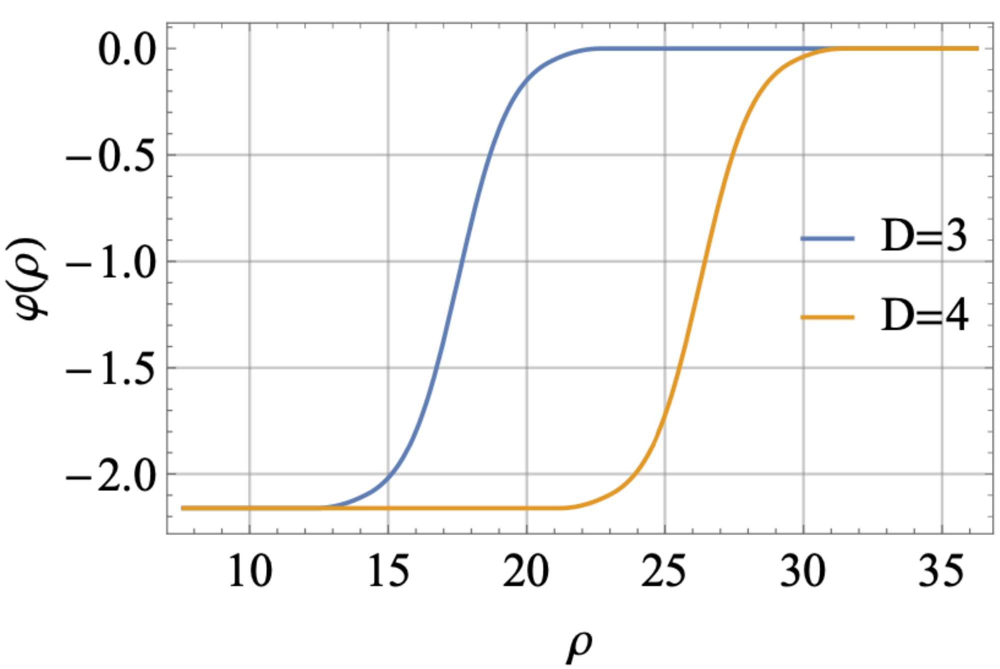
`“Dimension”`	sets the Eucludiean spacetime dimension, 3 or 4
`“FieldPoints”`	number of field points defines the segmentation
`“Gradient”`	one can pre-calculate the gradient, or make it numerical
`“Hessian”`	similar to the gradient, needed for multi-fields
`“MaxPath”`	limits the number of path iterations, typically small
`“MidFieldPoint”`	one can define a starting fixed point (e.g. the saddle)
`“PathTolerance” &` `“ActionTolerance”`	set a goal for the precision of the path variation and the Euclidean action

Output is a bundled container that can be easily accessed

`"Action"`	sets the Eucludiean spacetime dimension, 3 or 4
`"Bounce"`	number of field points defines the segmentation
`"Coefficients"`	one can pre-calculate the gradient, or make it numerical
`"Path"`	similar to the gradient, needed for multi-fields
`"Radii"`	limits the number of path iterations, typically small

In[9]:=
```
BouncePlot[{bf3,bf}, PlotLegends-> Placed[{"D=3","D=4"}, {Right,Center}]]
```

Out[9]=



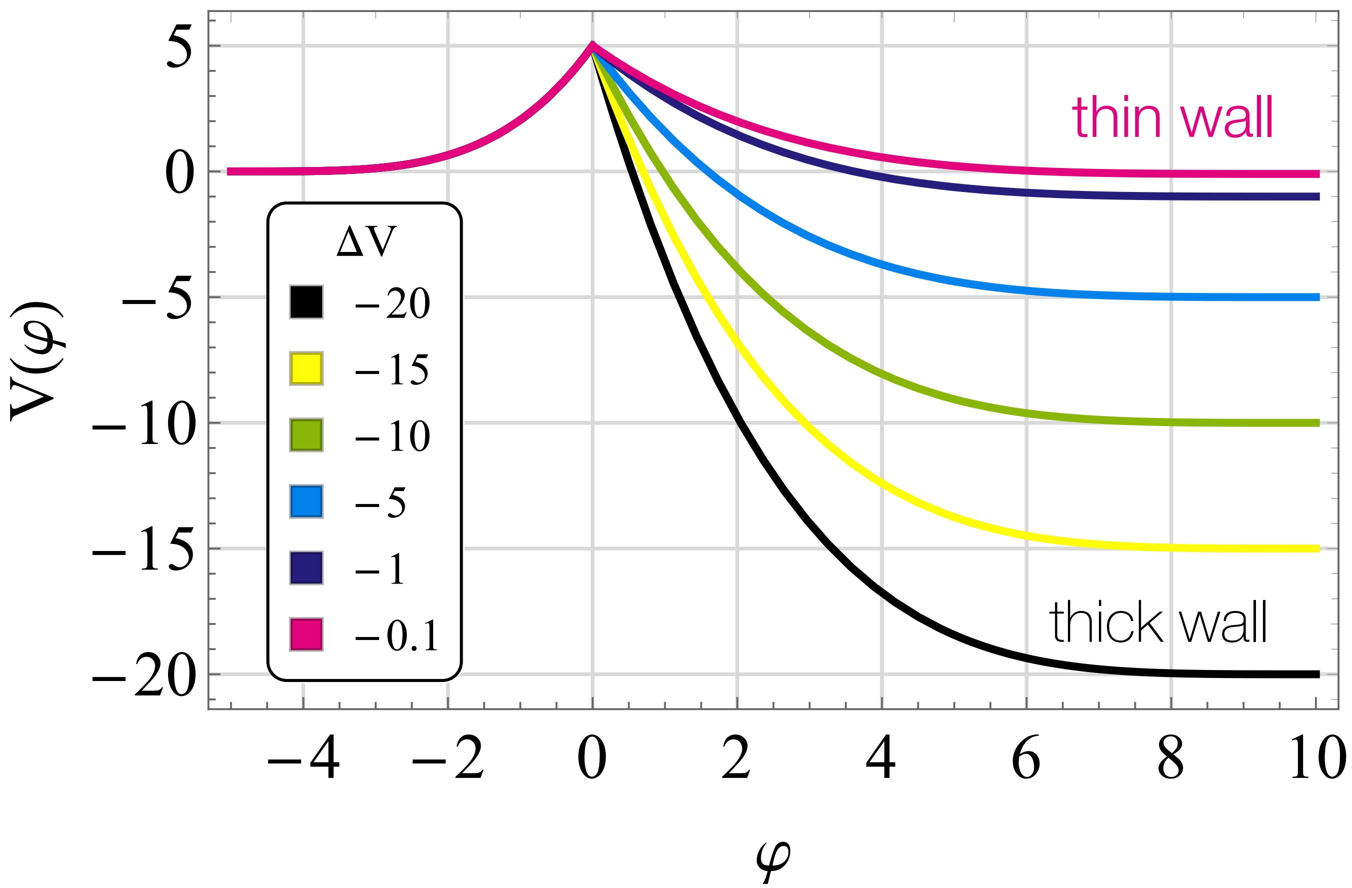
custom function for convenient plotting

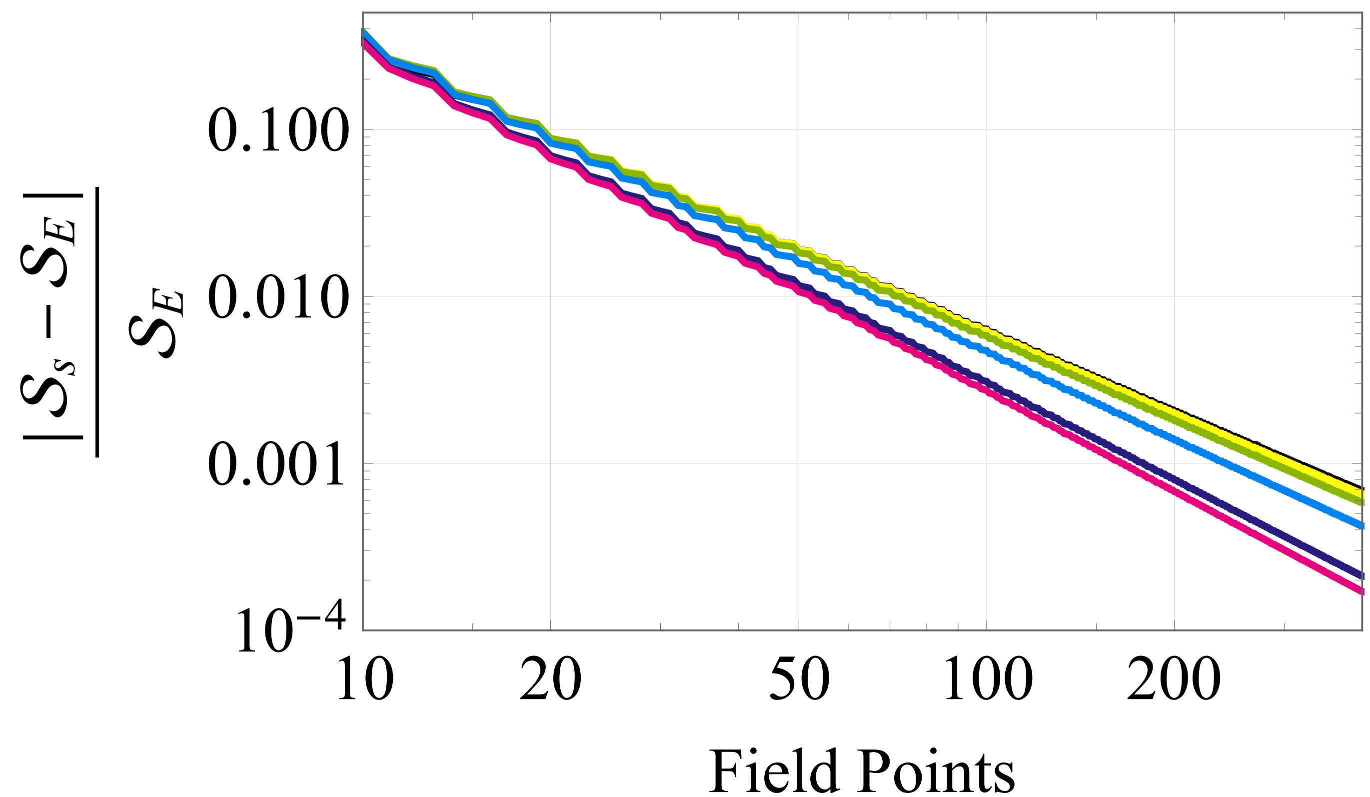
Bi-quartic

Other exact N=3 potentials, quartic-linear, quartic-quartic

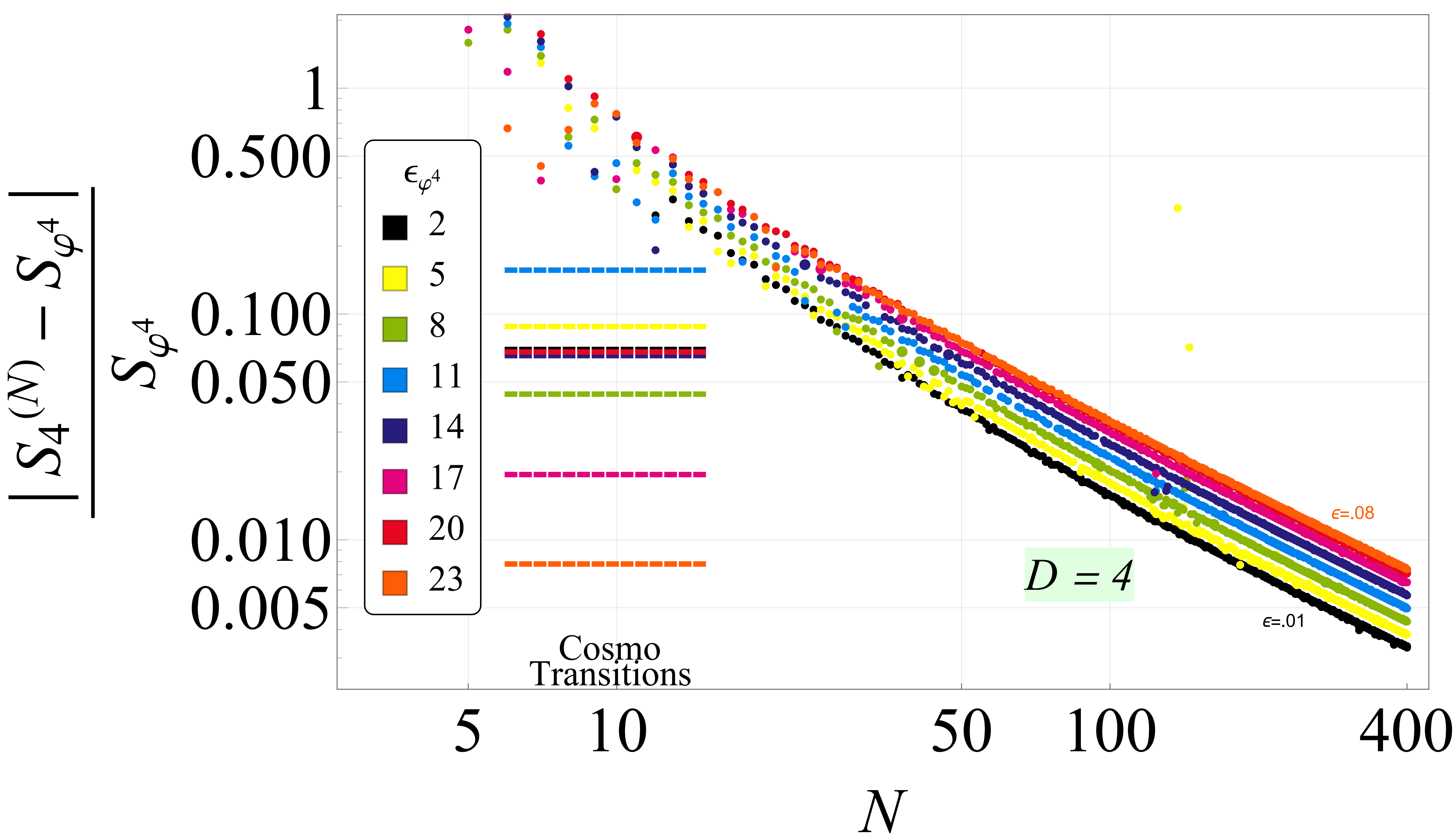
Dutta, Hector, Vaudrevange, Westphal '11

known exact solution, 'fair' comparison and test for the PB method

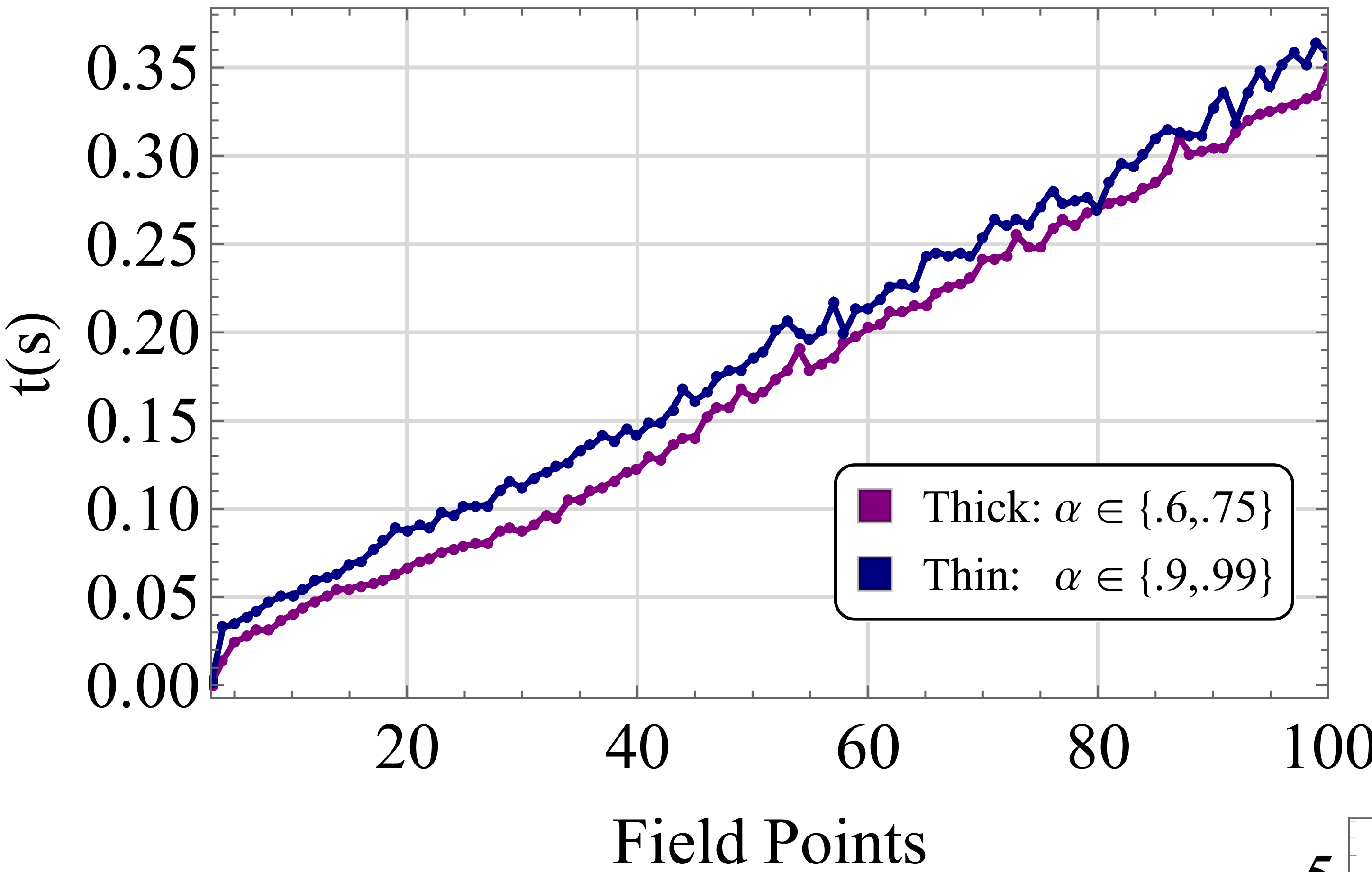




- CosmoTransitions — fails with the action, possible to repair by hand, precise from 20% to 0.5%
- AnyBubble — fails to compute
- Polygonal bounce — works smoothly with a bi-homogeneous segmentation



Time demand



Scales linearly by construction

Works in thin and thick regimes

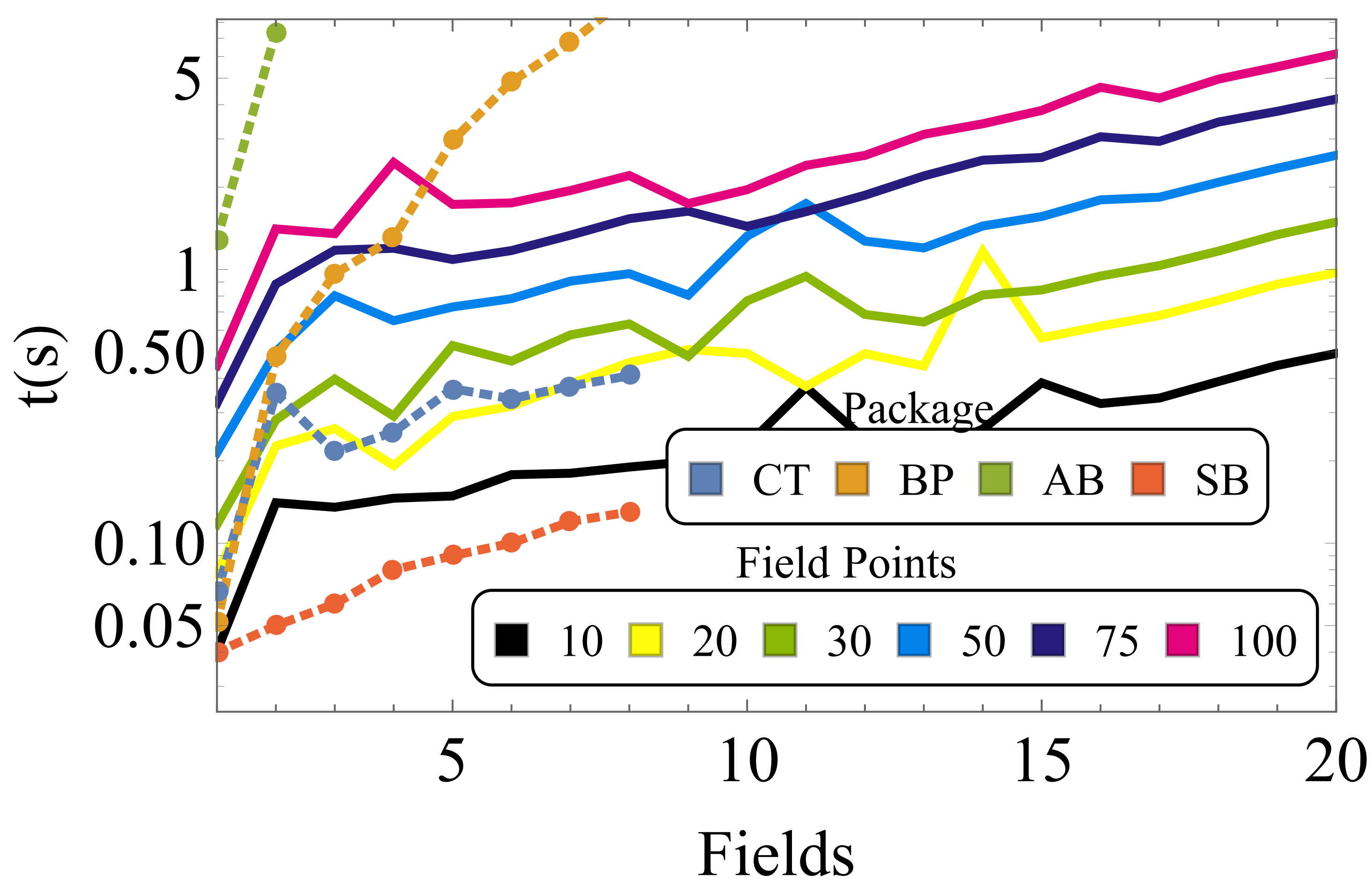
Tested up to 20 fields

CT - CosmoTransitions Wainwright '11

BP - BubbleProfiler Masoumi et al. '16

AB - AnyBubble Athron et al. '19

SB - SimpleBounce Sato '20

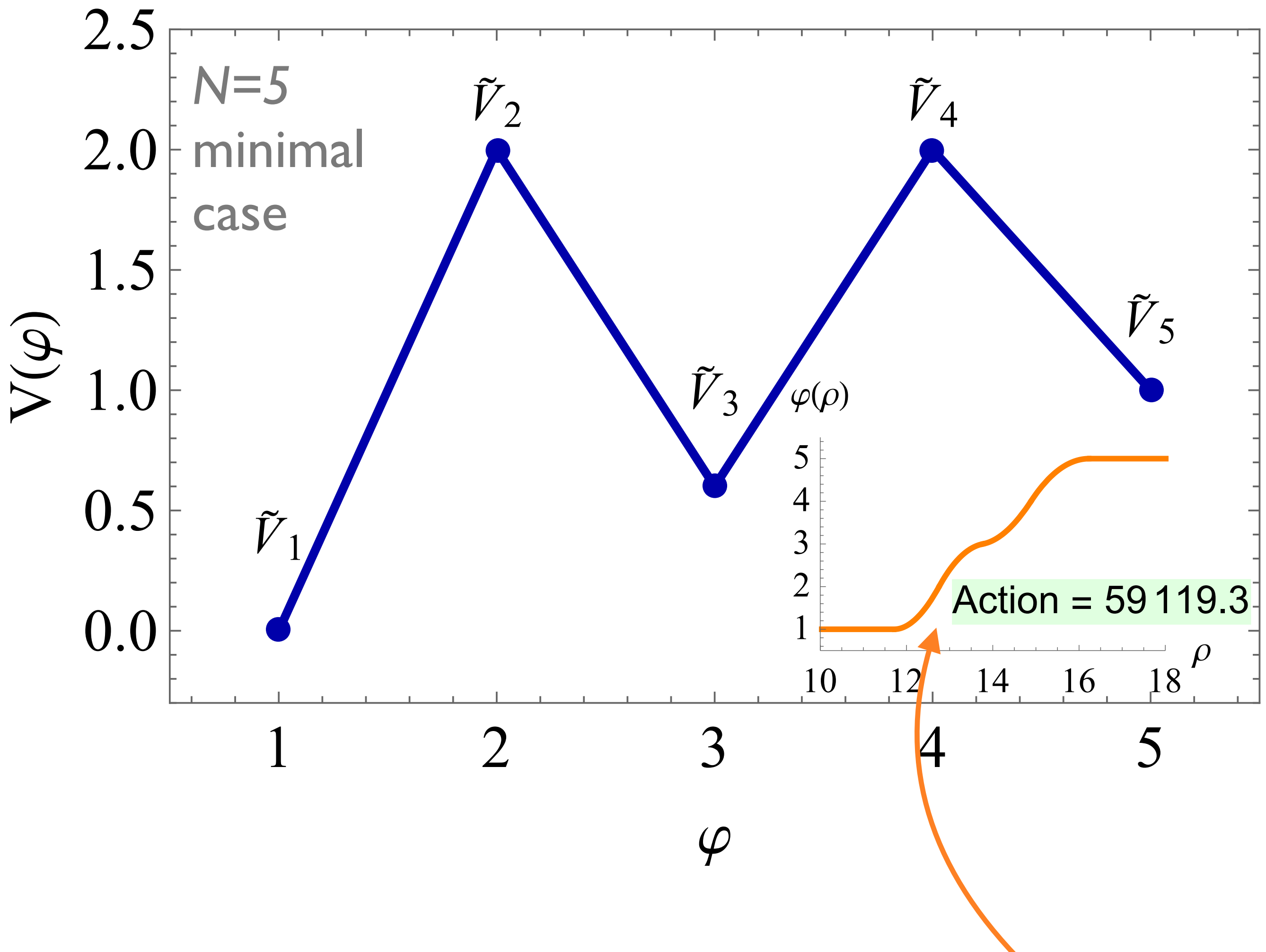


Disappearing instanton

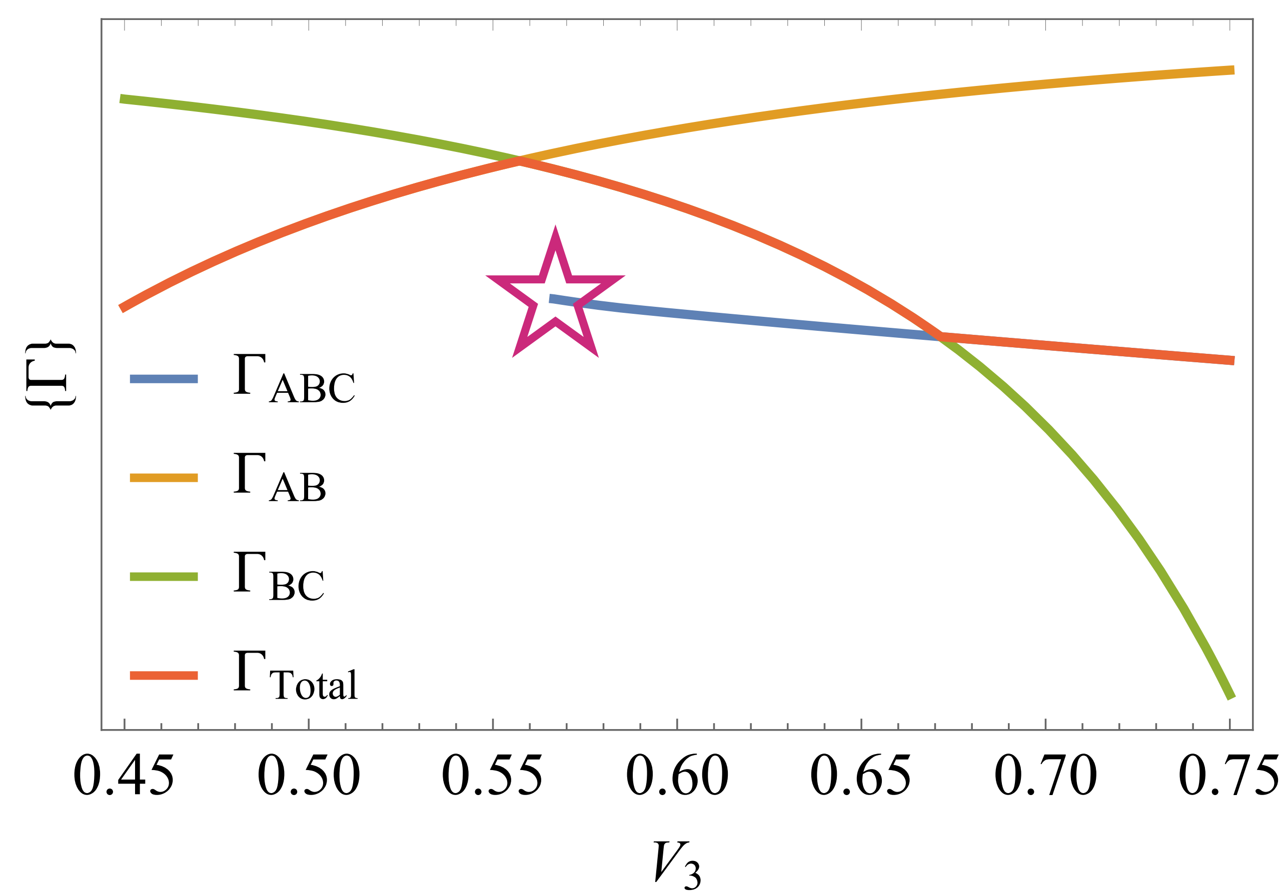
Intermediate minima, multi-step transitions — Dahlen, Brown '11

appear in theories with many fields, relaxion-type potentials

Q: which transition wins, direct tunneling or two subsequent transitions?



bubble with two walls



Direct tunneling impossible when intermediate minimum too low

Phase Transitions and

False Vacua are

fascinating and timely

Tunneling in QFT

11 January 2024 to 31 December 2030
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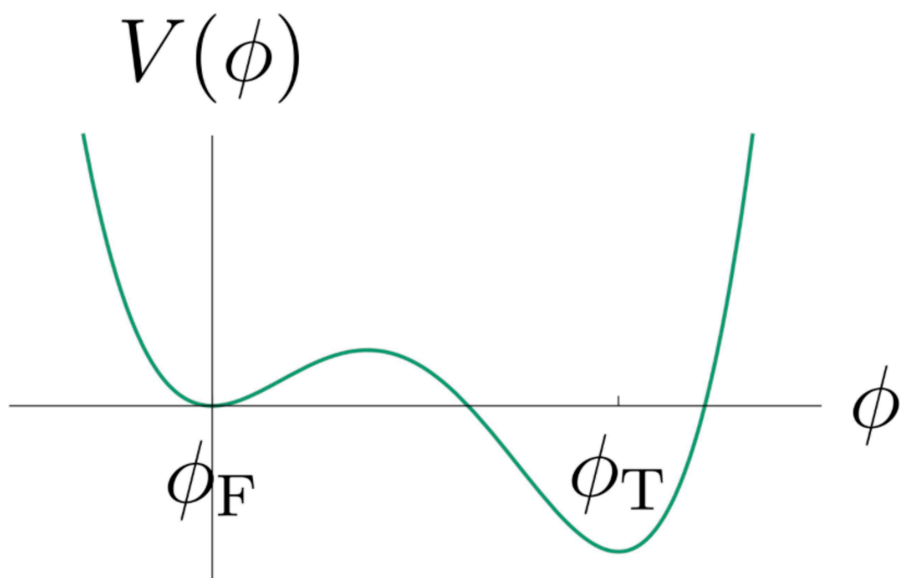
Seminar series devoted to tunneling in QFT

Monthly on Thursdays, usually @ 14:00 for Central Europe (CEST in European summer and CET in winter)

Whether it be vacuum stability, phase transitions, or analogue quantum systems, tunneling is part and parcel of quantum field theory. In this seminar series we explore new developments in our understanding of these phenomena.



Indico: https://indico.global/event/5685

Please register on this Indico page to receive Zoom joining instructions plus a reminder for each talk.

Youtube: https://www.youtube.com/@TunnelingQFT