Lecture 2
Neutron Star Equations of State - Basic Principles

Physics of Compact Star

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Thermodynamics

zero temperature limit

without nuclear burning

Hydrostatic equilibrium	$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$
Mass continuity	$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$
Radiative energy transport	$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3}$
Energy conservation	$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$

$$P = P(\rho, T, \text{composition})$$

$$\kappa = \kappa(\rho, T, \text{composition})$$

$$\epsilon = \epsilon(\rho, T, \text{composition})$$

Pressure

$$n = N/V$$

number density

Radiation pressure of photon & neutrino

$$P_{\rm rad} = \frac{1}{3} a T^4$$

Thermal (kinetic) pressure of ideal gas

$$P_{\rm kin} = nkT \qquad (PV = NkT)$$

Quantum degeneracy pressure of ideal gas

$$P_{\rm deg} = ? \qquad \text{(Pauli exclusion)}$$

Pressure from strong interactions

$$P_{\rm nuclear\ matter} = ?$$

Virial theorem

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = \int_0^R \left(-\frac{GM}{r^2}\rho\right) 4\pi r^3 dr$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = P \cdot 4\pi r^2 \Big|_0^R - \int_0^R 3P \cdot 4\pi r^2 dr$$

$$\int_0^R \left(-\frac{GM}{r}\right) 4\pi r^2 \rho dr$$

$$E_{\text{therm}} = \int_0^R \left(\frac{3}{2} nkT\right) 4\pi r^2 dr = \frac{1}{2}\int_0^R 3P \cdot 4\pi r^2 dr$$

$$-2E_{\text{therm}} = E_G$$

$$E_{\text{therm}} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\text{tot}} = E_G + E_{\text{therm}} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

Virial theorem

$$E_{\rm therm} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\rm tot} = E_G + E_{\rm therm} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

- Star forms by slow gravitational contraction
- Total energy becomes more negative
  - T & density increase as star loses energy

Fate of stars

T_c
- start nuclear burning
- thermal pressure dominates

$\rho_{\rm quantum}$
- reach quantum state
- degeneracy pressure dominates
- contraction stops and cools down

Polytropic Structure

n : polytropic index

ideal Fermi gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

cf)

Thermal pressure for an adiabatic process

$$pV^{\gamma} = \text{constant}$$

$$\gamma = \frac{c_p}{c_V}$$ ratio of specific heats

$$E = \frac{3}{2}\frac{N}{V}kT$$

$$P_{\text{kin}} = \frac{N}{V}kT$$

$$\gamma = \frac{5}{3}$$ mono-atomic gas

[Problem 1] The ideal Fermi gas equation of states of white dwarfs or neutron stars, in which quantum degeneracy pressure dominates, can be represented by polytropic form;

$$P_{\text{deg}} = K_\Gamma \rho^\Gamma = K_n \rho^{(n+1)/n}$$

where $K_\Gamma(K_n)$ and Γ are constants, ρ is density and n is called the polytropic index.

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

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a) For the comparison, consider an adiabatic expansion of an ideal monoatomic gas for which thermal (kinetic) pressure, $P_{\text{kin}} = (\rho/m)kT$, dominates and degeneracy pressure is negligible. Show that

$$TV^{\gamma-1} = \text{constant}, \qquad P_{\text{kin}}V^\gamma = \text{constant}, \qquad P_{\text{kin}} \propto \rho^\gamma$$

where $\gamma = c_P/c_V = 5/3$ is the ratio of specific heats (c_P: specific heat at constant pressure, c_V: specific heat at constant volume).

$$TV^{\gamma-1} = \text{constant}, \qquad P_{\text{kin}}V^{\gamma} = \text{constant}, \qquad P_{\text{kin}} \propto \rho^{\gamma}$$

$$E = \frac{3}{2}nkT$$

$$P_{\text{kin}} = nkT$$

$\Delta E_{th} = W + Q = -p\Delta V + Q$

단열과정 $\Delta E_{th} + p\Delta V = 0$ $(Q=0)$

E_{th} 대입 ⇓ $\Delta E_{th} = nC_V\Delta T$

$nC_V dT + p\,dV = 0$

dT 대입 ⇓ $pV = nRT$

$p\,dV + dp\,V = nR\,dT$

$\frac{C_V}{R}(p\,dV + dp\,V) + p\,dV = 0$

⇓ $\frac{C_V+R}{R}p\,dV + C_V V\,dp = 0$

$\frac{dp}{p} + \frac{C_V+R}{C_V}\frac{dV}{V} = 0$

$\frac{dp}{p} + \frac{C_V+R}{C_V}\frac{dV}{V} = 0$

⇓ $\gamma \equiv \frac{C_V+R}{C_V} = \frac{C_P}{C_V}$

$\frac{dp}{p} + \gamma\frac{dV}{V} = 0$

⇓ $\ln p + \gamma\ln V = C$

$\ln p + \ln V^{\gamma} = \ln pV^{\gamma} =$ 상수

$pV^{\gamma} =$ 상수

$$\gamma = \frac{5}{3}$$ mono-atomic gas

Ideal Degenerate Electron Gas

$$\Delta x \Delta p_x > h \qquad d^3 p dV > h^3$$

$$\rho_{\text{quantum}} \approx \frac{m_p}{(\lambda_e/2)^3} = \frac{8 m_p (3 m_e k T)^{3/2}}{h^3}$$

$$\rho_{\text{quantum}}(\text{sun, center}) \approx 640 \text{ g/cm}^3$$

$$\rho_{\text{quantum}}(T = 10^8 \text{ K}) \approx 11,000 \text{ g/cm}^3$$

$$n_e = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_f^3 = Z n_+ = Z \frac{\rho}{A m_p}$$

n_e, n_+ number density

white dwarfs

$$\rho_{\text{WD}} \sim 10^6 \text{ g/cm}^3$$

Workout Problem

$$\left(20°\text{C} = 293K \approx \frac{1}{40}\text{eV}\right)$$

Without nuclear reaction, star will contract to the quantum limit.

de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \approx \frac{h}{\sqrt{3mkT}}$$

$$E \sim \frac{3}{2}kT$$

* electron will reach quantum domain first

Check.

If we assume inter-particle separation $\sim \frac{\lambda}{2}$

→ quantum domain

$$\rho_{quantum} \approx \frac{m_p}{(\lambda/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}$$

With $T = 15 \times 10^6$ K

$$\rho_{quantum} \approx 640 \text{ g/cm}^3$$

$$\rho_{center,\,sun} \sim 150 \text{ g/cm}^3$$

↑ classical.

Workout Problem

b) For ideal Fermi gas, in the zero temperature limit, show that the number density of gas (n_g) and Fermi momentum (p_F) are related by

$$n_g = g \times \frac{2\pi}{3h^3} p_F^3$$

where g is the degeneracy.

Consider simple case $(T=0)$

isotropic distribution

$d^3p \rightarrow 4\pi p^2 dp$

Fermi momentum

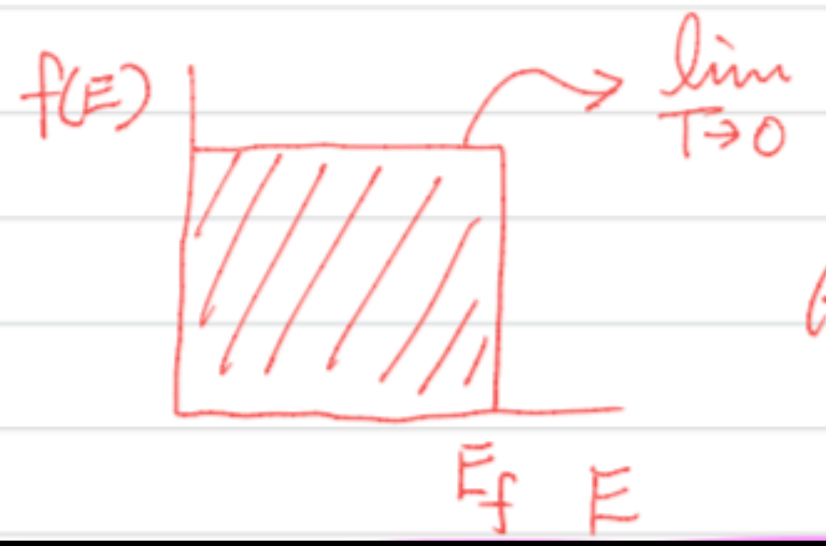
$$dN(p)dp = 2\times\frac{d^3p\,dV}{h^3} = \begin{cases} 2\times\frac{4\pi p^2 dp\, dV}{h^3} & \text{if } |\vec{p}|\le p_F \\ 0 & \text{if } |\vec{p}|> p_F \end{cases}$$

$$n_e(p)dp = \frac{8\pi p^2 dp}{h^3}$$

$$n_e = \int_0^{p_F} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

Normal gas $P(T=0)=0$.

Quantum gas $P(T=0)\neq 0$ from $\Delta x^3 \Delta p^3 > h^3$



All states with $E \le E_F$ are occupied (degenerate)

Typical white dwarfs

$$\rho_{\rm WD} \sim 10^6 \ {\rm g/cm^3}$$

$$T_{\rm WD} \sim 10^7 \ {\rm K}$$

Quantum degeneracy pressure dominates : *thermal pressure negligible*

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$P_{\rm deg} \approx 3 \times 10^{22} {\rm dyne/cm^2}$$

$$P_{\rm th} \approx 2 \times 10^{20} {\rm dyne/cm^2}$$

Degeneracy Pressure / Polytropic Structure

n : polytropic index

ideal Fermi gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

Non-relativistic

$$\gamma = \frac{5}{3} \qquad n = \frac{3}{2} \qquad P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}$$

Ultra-relativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

$$P_e \propto \int_0^{\infty} n_e(p) p v dp \propto \int_0^{p_F} \frac{p^4}{\sqrt{p^2 + m^2}} dp$$

p : momentum transfer

v : number of particles

$p = \gamma m v$

Workout Problem

Non-relativistic

c) In the non-relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

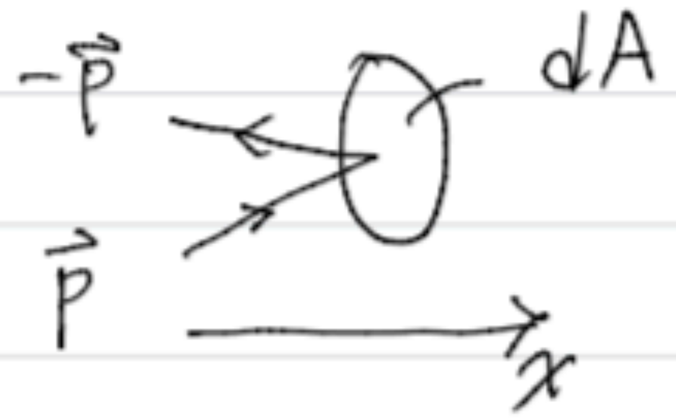
Ultra-relativistic

d) In the full relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

Degeneracy Pressure



$$\Delta p_x = 2p_x$$

$$v_x = \frac{dx}{dt}$$

change of momentum

\# of particles

$$\frac{dF_x}{dA} = \frac{2p_x}{dA\,dt} = \frac{2p_x}{dA\,(dx/v_x)} = \frac{2p_x v_x}{dV}$$

↑ The force per unit area due to each collision.

$$P = \int_0^\infty \frac{dN(p)}{2} \frac{2p_x v_x}{dV}\, dp$$

↑ only half of electrons are moving $+x$ at any given time.

$$p_x v_x = \frac{1}{3} p v.$$

$$\begin{cases} v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{3 v_x^2} \\ p = \sqrt{3 p_x^2} \\ vp = 3\, p_x v_x \end{cases}$$

$$P_e = \frac{1}{3} \int_0^\infty n_e(p)\, p v\, dp$$

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} v\, p^3\, dp$$

Degeneracy Pressure

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} v\, p^3 dp$$

$$p = \gamma m v \quad , \quad E = \gamma m c^2$$

$$v = \frac{p}{m\gamma} = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}}$$

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}}\, dp$$

$$n_e = \frac{X\rho}{m_H} + \frac{(1-X)\rho}{2m_H} = \frac{\rho}{2m_H}(1+X)$$

$$n_e = \frac{\rho}{\mu_e m_H} \quad ; \quad \mu_e = \frac{2}{1+X}$$ (mean molecular weight of electron)

from

$$n_e = \frac{8\pi}{3h^3} p_F^3$$

$$p_F = \left(\frac{3h^3 \rho}{8\pi \mu_e m_H}\right)^{1/3}$$

Workout Problem

non-relativistic

c) In the non-relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} \, dp$$

i) non-relativistic $\quad \sqrt{p^2c^2 + m_e^2c^4} \approx m_e c^2$

$$P = \frac{8\pi}{15\, h^3 m_e} p_F^5$$

$$P = K_1 \rho^{5/3}$$

$$K_1 = \frac{3^{2/3}}{20\,\pi^{2/3}} \frac{h^2}{m_e m_H^{5/3} \mu_e^{5/3}} = \frac{1.00 \times 10^7}{\mu_e^{5/3}}$$

Workout Problem

ultra-relativistic

d) In the full relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}}\, dp$$

ii) fully relativistic $\sqrt{p^2c^2 + m_e^2c^4} \approx pc$

$$P = \frac{2\pi c}{3h^3} p_F^4$$

$$P = K_2 \rho^{4/3}$$

$$K_2 = \frac{3^{1/3}}{8\pi^{1/3}} \frac{hc}{m_H^{4/3} \mu_e^{4/3}} = \frac{1.24 \times 10^{10}}{\mu_e^{4/3}}$$

Lane-Emden Equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

$$P = K\rho^\gamma$$

$$\rho = \rho_c \theta^n$$

$$r = a\xi$$

$$a = \sqrt{\frac{(n+1)K_n\rho_c^{(1-n)/n}}{4\pi G}}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

$$\rho = \rho_c \theta^n$$

center $\theta(0) = 1, \quad \theta'(0) = 0$

surface $\theta(\xi_1) = 0$ → $P = \rho = 0$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

$$R = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}\, \xi_1$$

Constants of Lane-Emden equation

$$r = a\xi$$

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

Stellar Structure by S. Chandrasekhar

Polytropic Structure

n : polytropic index

ideal gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

Ultra-relativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

Chandrasekhar limit

$$\mu_e = A/Z \approx 2 \text{ for He, C, O,..}$$

Ultra-relativistic $\gamma = \frac{4}{3}$ $n = 3$

$$M = 1.457 \left(\frac{2}{\mu_e}\right)^2 M_\odot$$

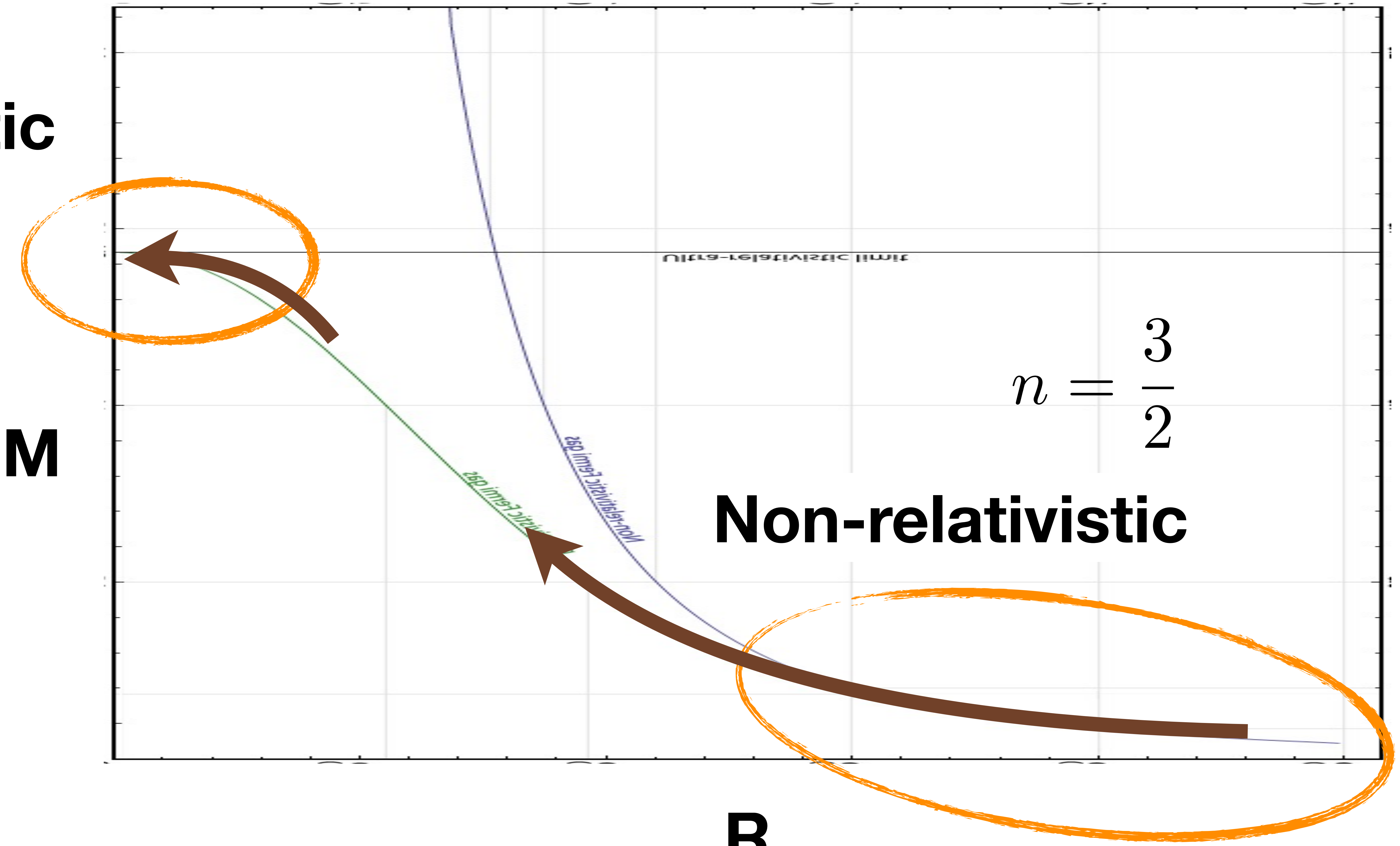
$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/3} \left(\frac{2}{\mu_e}\right)^{2/3} \text{ km}$$

Mass is independent of central density

White Dwarfs - electron degeneracy pressure

$n = 3$

Ultra relativistic



M

$n = \frac{3}{2}$

Non-relativistic

R

White Dwarfs

$\mu_e = A/Z \approx 2$ for He, C, O, ..

Non-relativistic $\gamma = \frac{5}{3} \quad n = \frac{3}{2}$

$$R = 1.122 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/6} \left(\frac{2}{\mu_e} \right)^{5/6} \text{ km}$$

$$M = 0.7011 \left(\frac{R}{10^4 \text{ km}} \right)^{-3} \left(\frac{2}{\mu_e} \right)^5 M_\odot$$

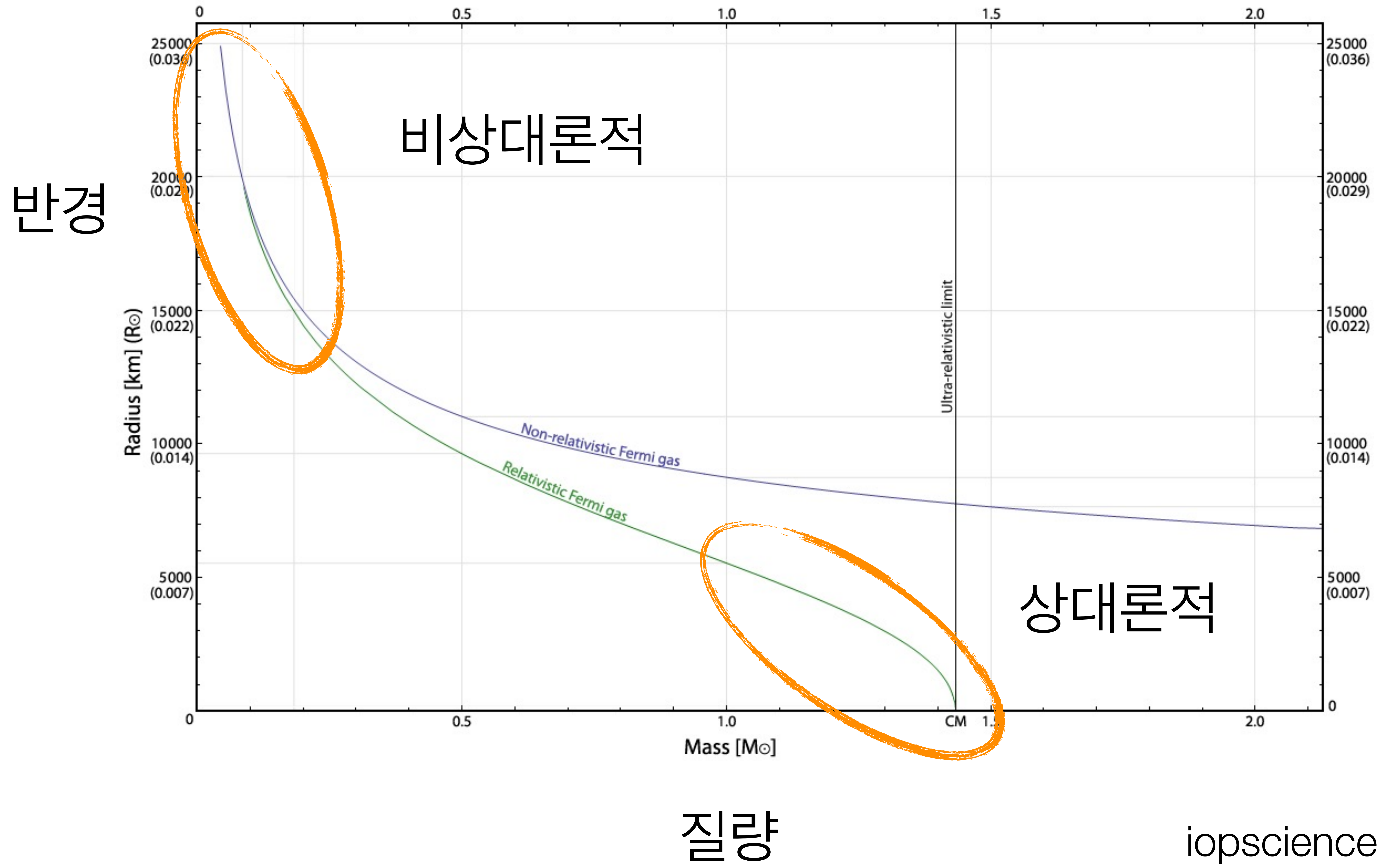
Ultra-relativistic $\gamma = \frac{4}{3} \quad n = 3$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/3} \left(\frac{2}{\mu_e} \right)^{2/3} \text{ km}$$

$$M = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_\odot$$

independent of central density

White Dwarfs



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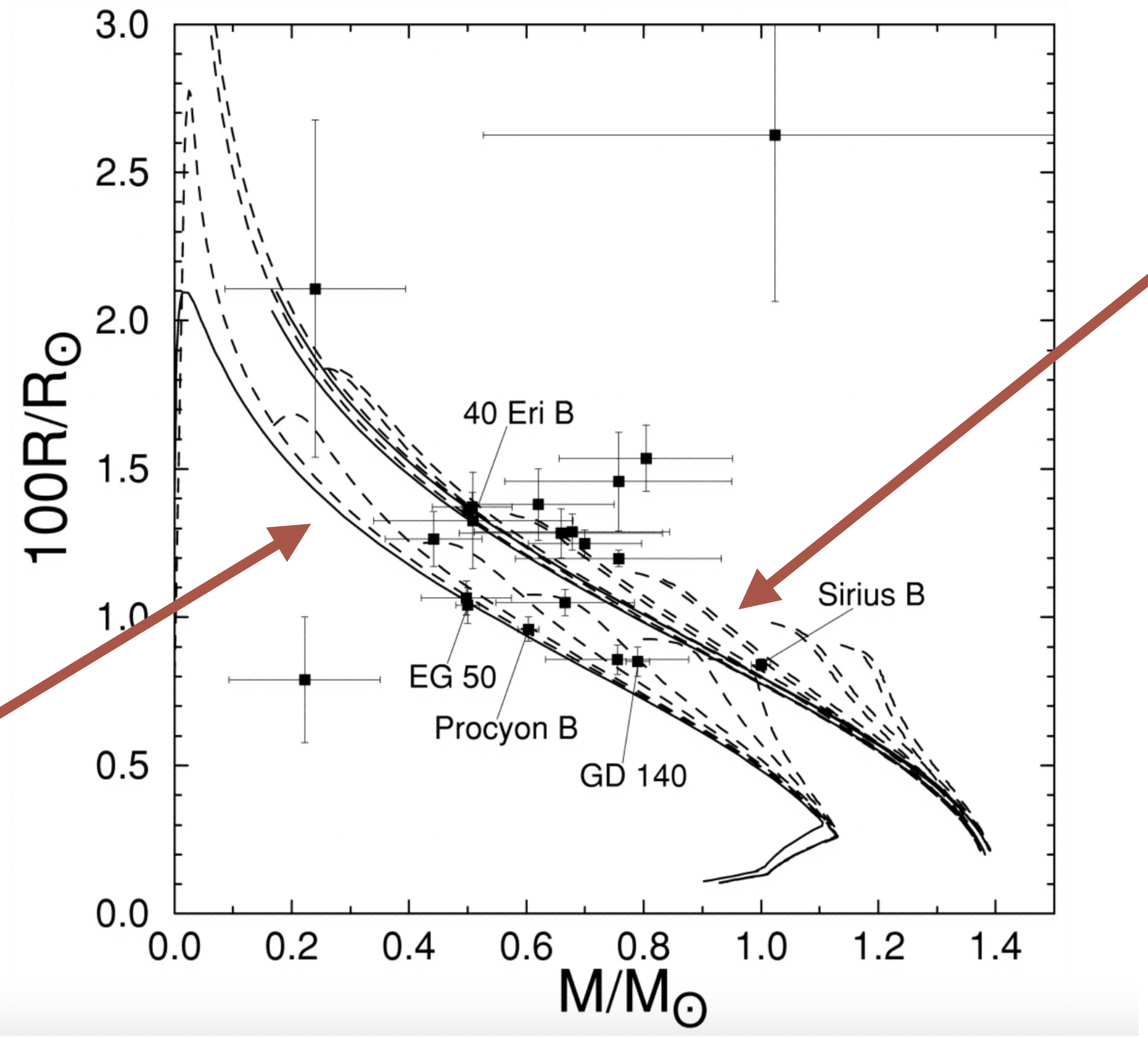
White Dwarfs

ApJ 530, 949 (2000)

$^{4}\mathrm{He},\ ^{12}\mathrm{C}$

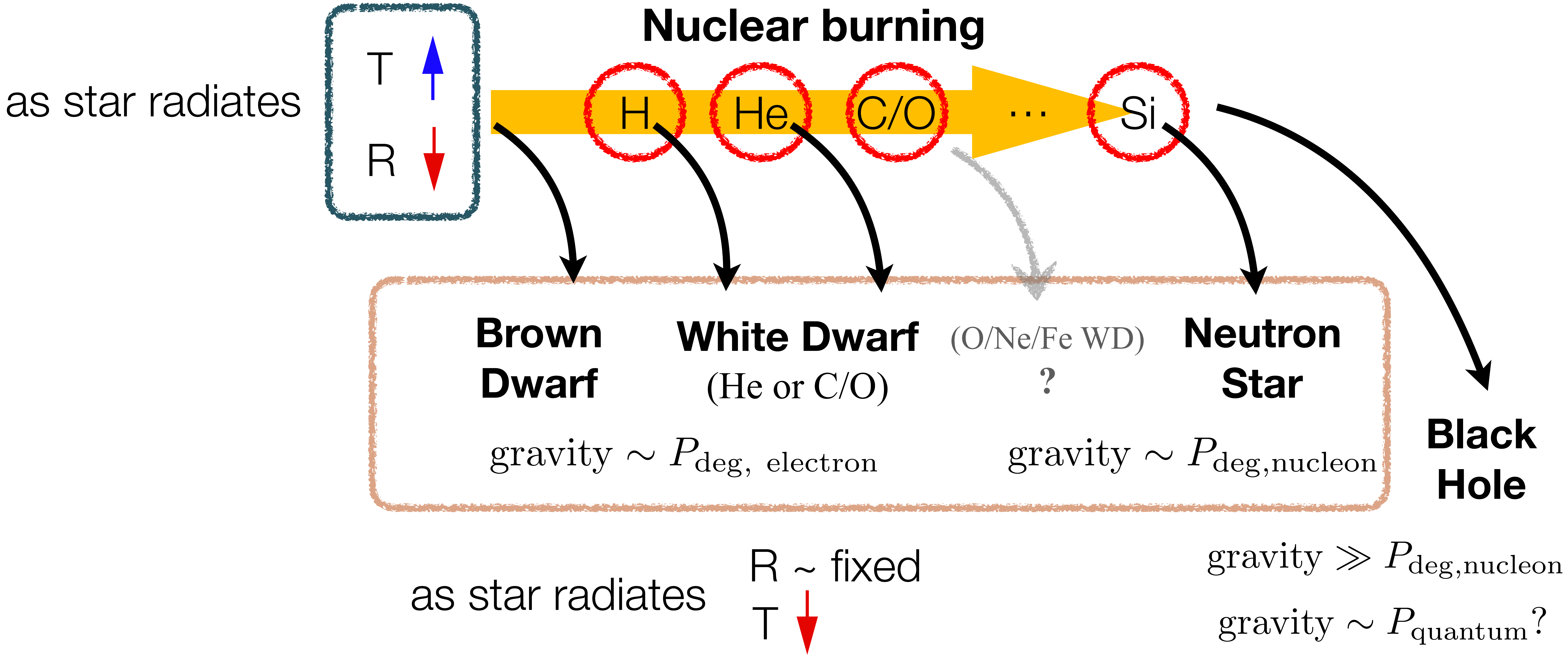


$^{56}\mathrm{Fe}$

Q) What about neutron star?

Nucleon degeneracy pressure

Evolution of star



What's inside black hole ?

quantum gravity

$$\text{gravity} \sim P_{\text{quantum}}?$$

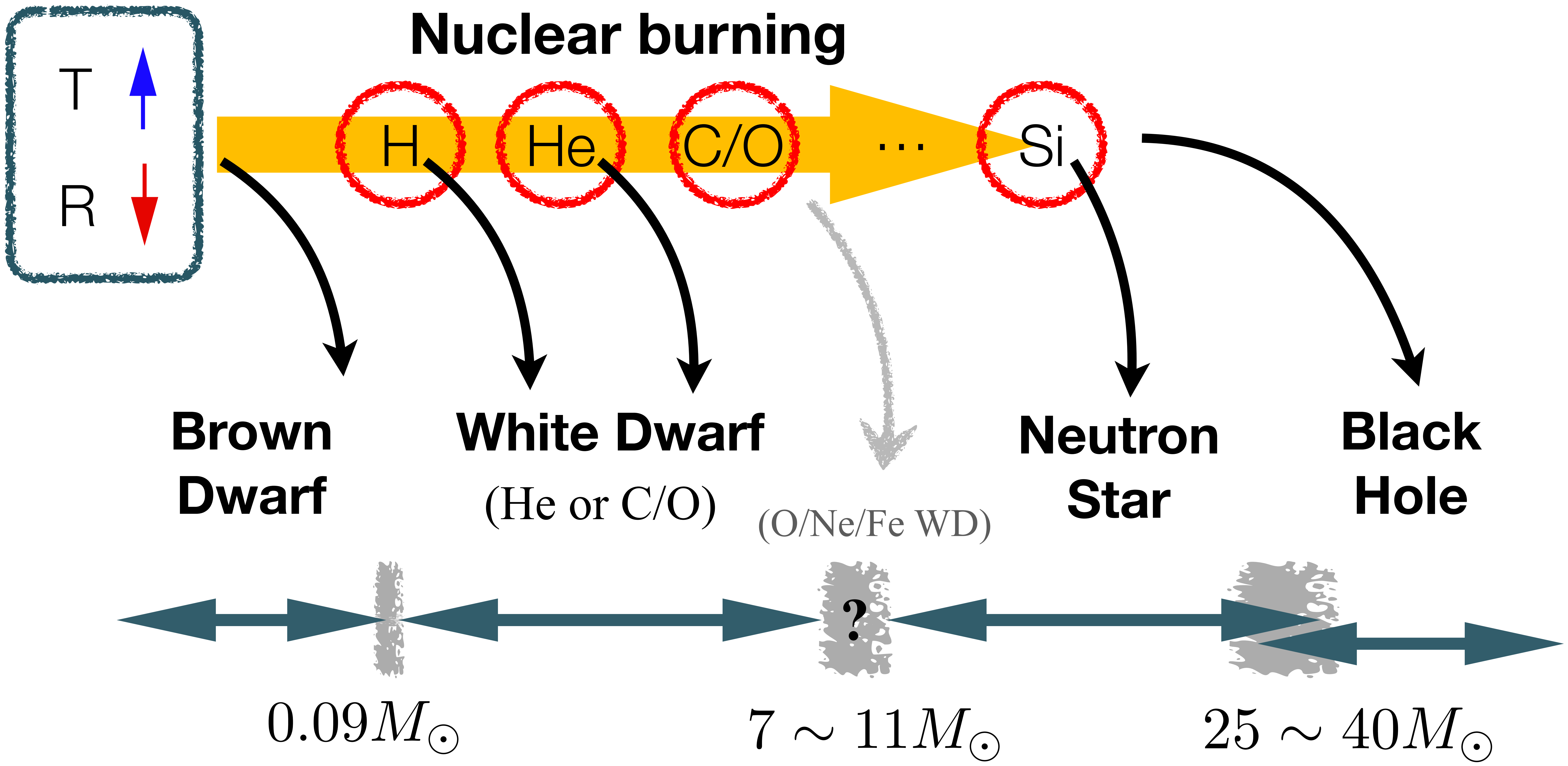
$$\frac{GMm}{l} \sim \frac{G}{l}\frac{E}{c^2}m \sim \frac{G}{lc^2}\frac{\hbar c}{l}m = \frac{G\hbar}{l^2 c}m \sim mc^2$$

$$M \sim \frac{E}{c^2} \qquad E \sim \frac{hc}{\lambda} \sim \frac{hc}{2\pi l} = \frac{\hbar c}{l} \qquad l \sim \sqrt{\frac{G\hbar}{c^3}}$$

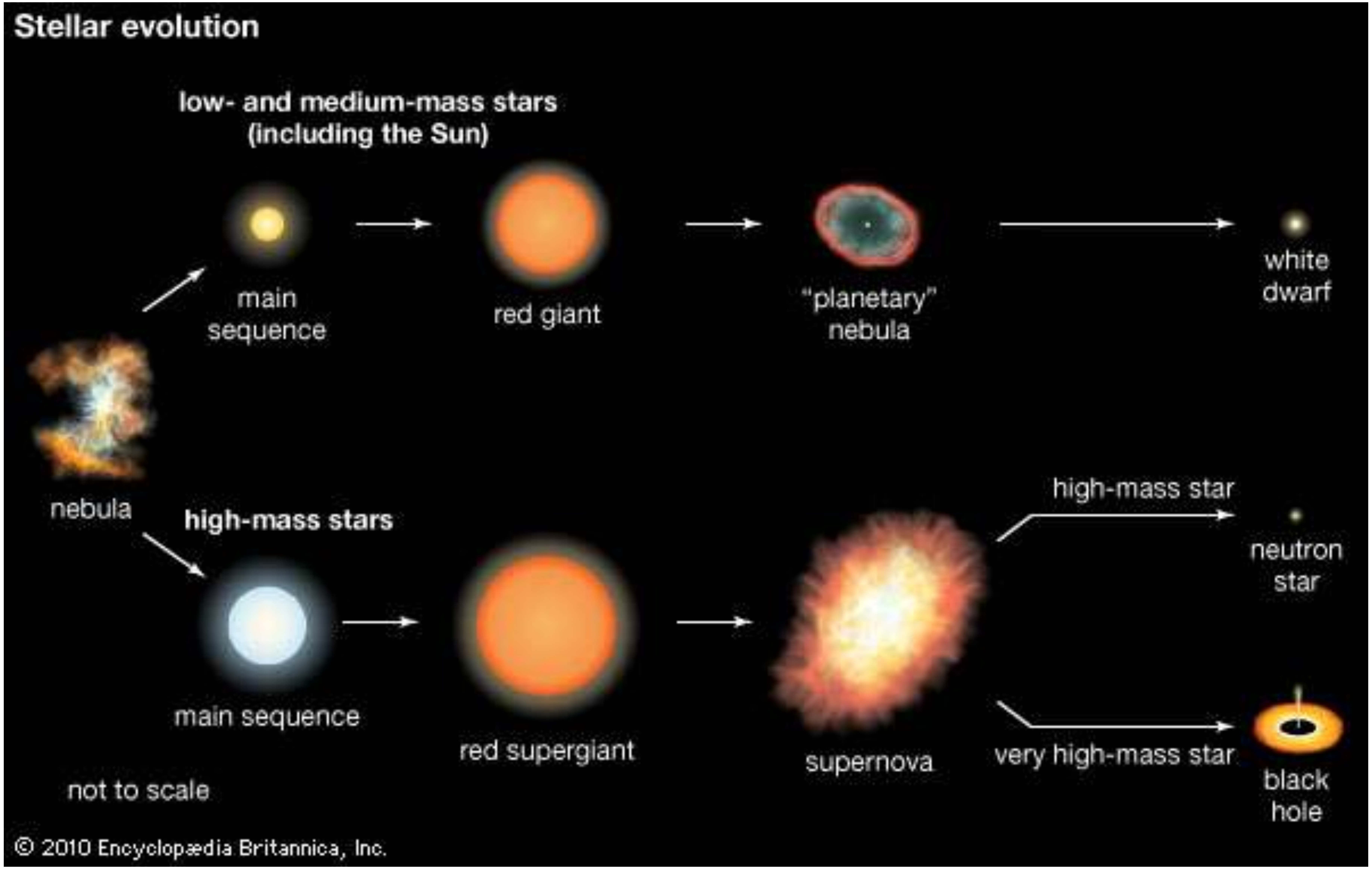
$$l_{\text{Planck}} \sim 10^{-33}\ \text{cm}$$

$$t_{\text{Planck}} = \frac{l_{\text{Planck}}}{c} \sim 10^{-43}\ \text{s}$$

Initial mass of progenitor star (with solar metallicity)



ZAMS (zero age main sequence) mass


Stellar evolution
low- and medium-mass stars
(including the Sun)
main
sequence
red giant
"planetary"
nebula
white
dwarf
nebula
high-mass stars
main sequence
red supergiant
supernova
high-mass star
neutron
star
very high-mass star
black
hole
not to scale
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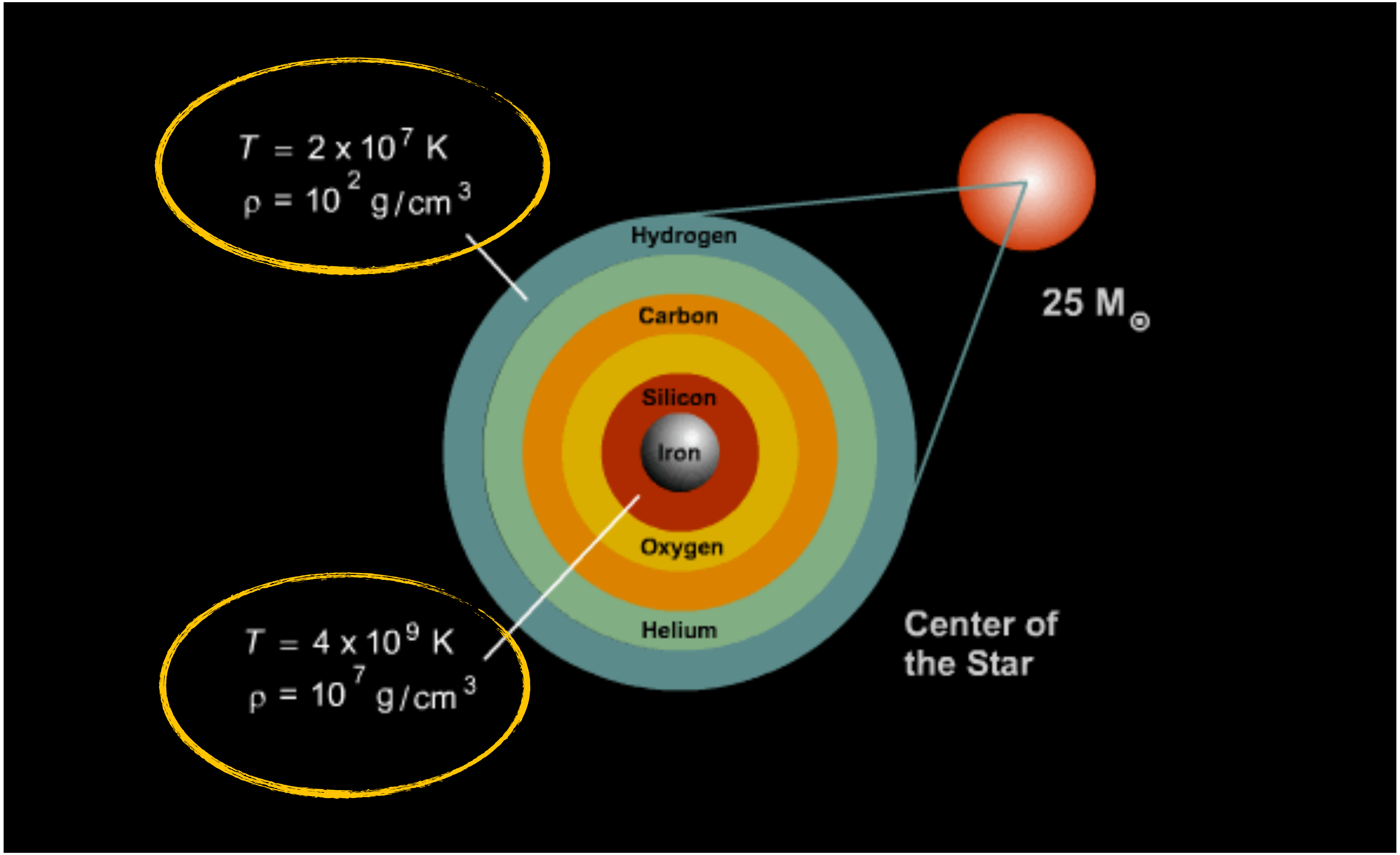


$T = 2 \times 10^7$ K
$\rho = 10^2$ g/cm^3
Hydrogen
Carbon
Silicon
Iron
Oxygen
Helium
$T = 4 \times 10^9$ K
$\rho = 10^7$ g/cm^3
25 $M_\odot$
Center of
the Star

Neutron Star Formation from Fe Core Collapse

When Fe core reaches Chandrasekhar Limit

(n : neutron)

Nuclear Photodisintegration

$$\gamma + {}^{56}\mathrm{Fe} \to 13\ {}^{4}\mathrm{He} + 4\ \mathrm{n}\ (124\ \mathrm{MeV\ is\ consumed})$$

$$\gamma + {}^{4}\mathrm{He} \to 2\ \mathrm{p} + 2\ \mathrm{n}\ (28.3\ \mathrm{MeV\ is\ consumed})$$

$$E_{\mathrm{consumed}}(10^{57}\mathrm{p}) \sim 1.4 \times 10^{52}\ \mathrm{erg} \sim 10^{11}\mathrm{year\ with}\ L_{\odot}$$

Neutralisation

$$\mathrm{e}^- + \mathrm{p} \to \mathrm{n} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^- + {}^{56}\mathrm{Fe} \to {}^{56}\mathrm{Mn} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^- + {}^{\mathrm{Mn}}\mathrm{Fe} \to {}^{56}\mathrm{Cr} + \nu_{\mathrm{e}}$$

core collapse → NS + Supernova

$$R_{\mathrm{Fe}} \sim 1500\mathrm{km}$$

$$R_{\mathrm{NS}} \sim \mathcal{O}(10\mathrm{km})$$

$$\tau_{\mathrm{fall}} \sim \mathrm{a\ few\ seconds}$$

Properties of Neutron Star

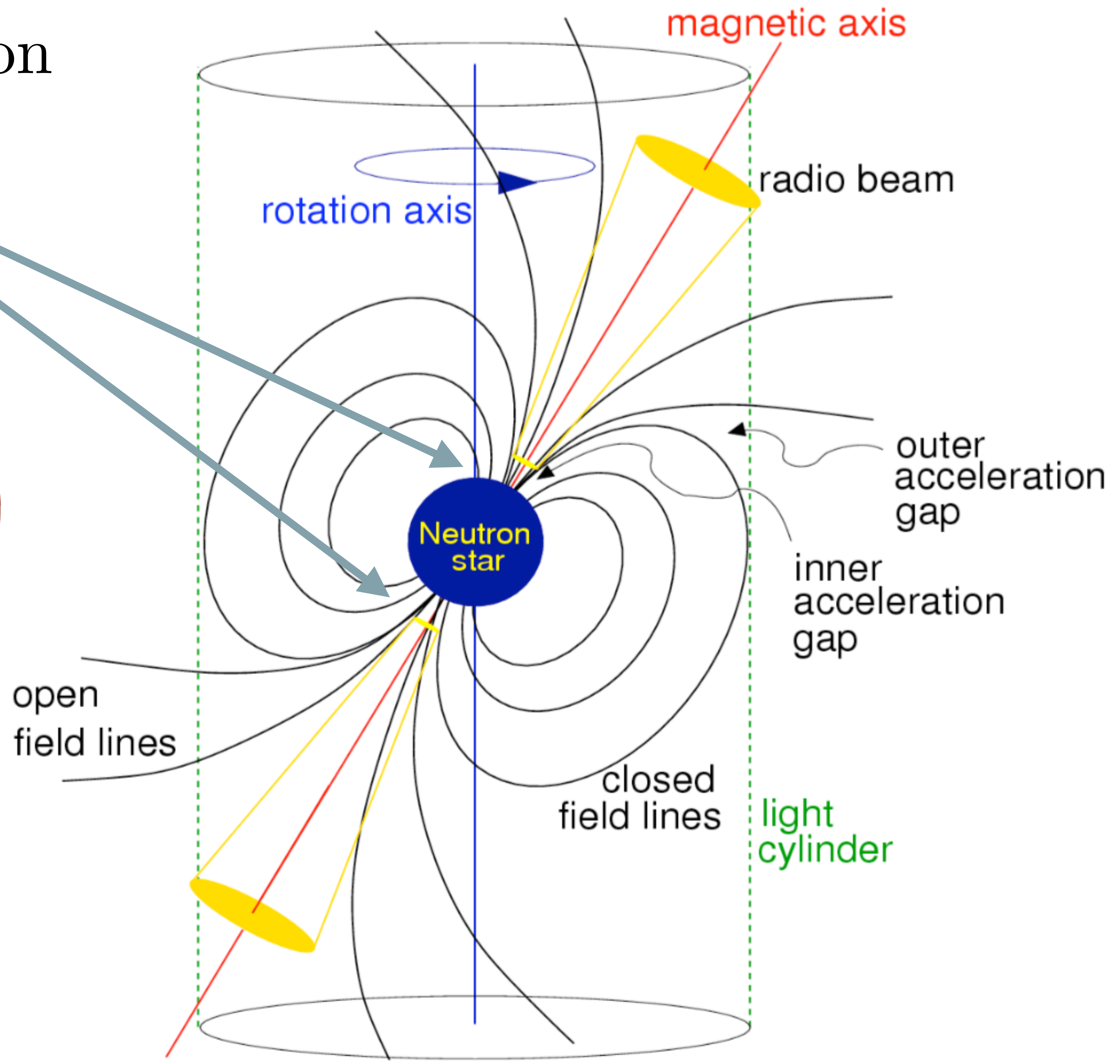
e^+e^- pair creation

Pulsar

$$M = 1.5 \sim 2.0 M_\odot$$
$$R = 10 \sim 15 \text{ km}$$
$$A \sim 10^{57} \text{ nucleons}$$

$$\rho_{\text{center}} \approx \text{several} \times \rho_0$$

$$n_0 \approx 0.16 \text{ fm}^{-3}$$
$$\approx 1.6 \times 10^{44} \text{m}^{-3}$$
$$\rho_0 \approx 2.04 \times 10^{17} \text{kg} \cdot \text{m}^{-3}$$



Educated Guesses

white dwarfs

$$\rho_{\rm WD} \sim 10^6 \ {\rm g/cm^3}$$

$$T_{\rm WD} \sim 10^7 \ {\rm K}$$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \ {\rm g \ cm^{-3}}} \right)^{-1/3} \left(\frac{2}{\mu_e} \right)^{2/3} \ {\rm km}$$

$$M = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_\odot$$

Quantum degeneracy pressure of protons/neutrons

neutron stars

$$\mu_e = A/Z \to \mu_n = 1$$

$$R_{\rm NS} \approx 2.3 \times 10^9 \ {\rm cm} \left(\frac{m_e}{m_n} \right) \left(\frac{1}{\mu_n} \right)^{5/3} \left(\frac{M}{M_\odot} \right)^{-1/3} \approx 14 \ {\rm km} \left(\frac{M}{1.4 M_\odot} \right)^{-1/3}$$

$$M_{\rm NS,Chan} \approx 0.2 \left(\frac{2}{\mu_n} \right)^2 \left(\frac{hc}{G m_p^2} \right)^{3/2} m_p \approx 4 \times 1.4 M_\odot = 5.6 M_\odot$$

$$\rho \sim 10^{14} {\rm g/cm^3}$$

General Relativity

hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \qquad \frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{Mc^2}\right)\left(1-\frac{2GM}{rc^2}\right)^{-1}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \qquad \frac{dM}{dr} = 4\pi r^2\left(\frac{\epsilon}{c^2}\right)$$

include all energy sources

physics of dense nuclear matter (strong interaction)

Without general relativity

n : polytropic index

ideal gas $p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

nonrelativistic

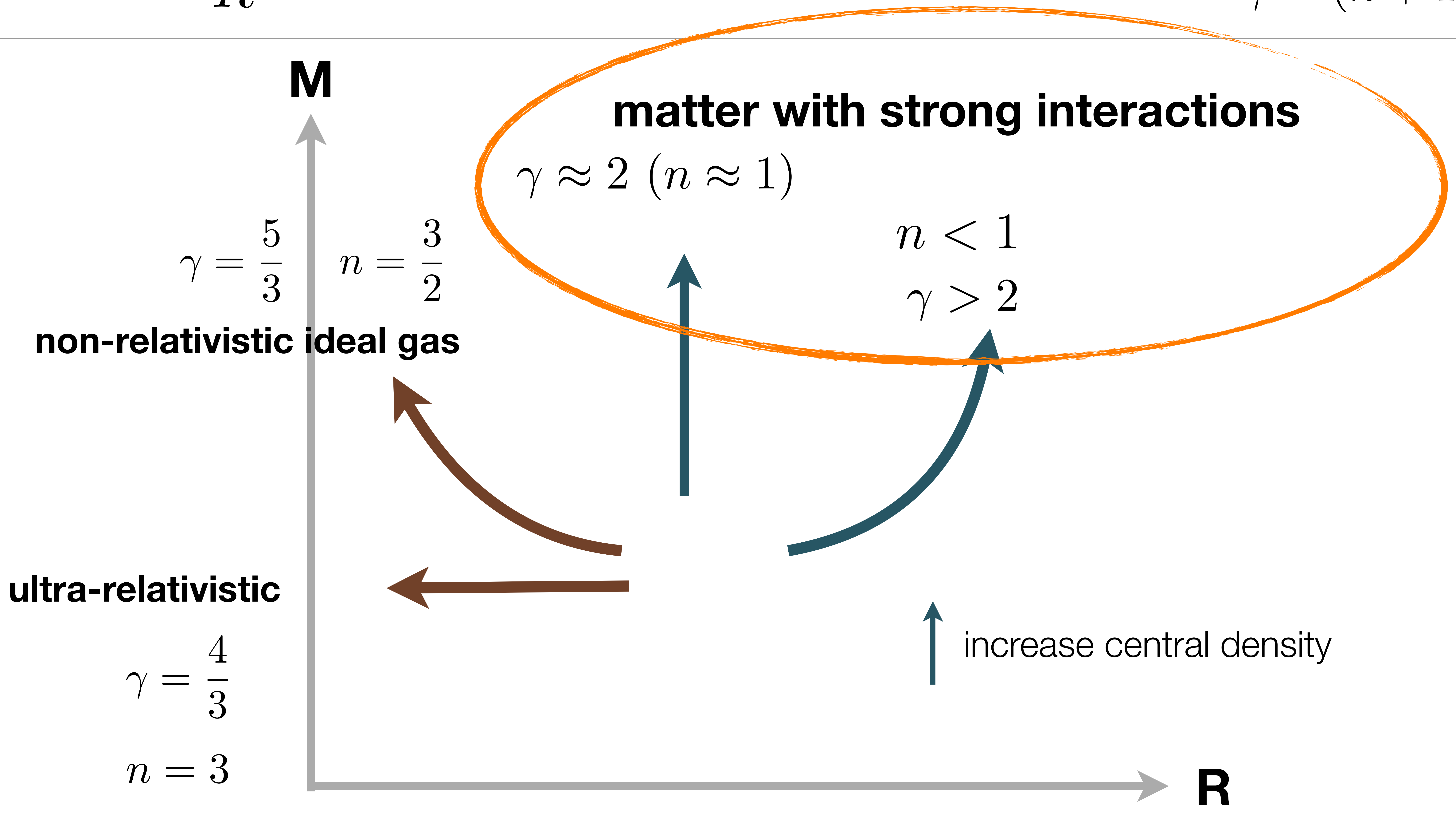
$$\gamma = \frac{5}{3} \qquad n = \frac{3}{2} \qquad P_N = \left(\frac{3}{\pi}\right)^{2/3} \left(\frac{h^2}{20 m_N}\right) \left(\frac{\rho}{m_N}\right)^{5/3}$$

ultrarelativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_N = \left(\frac{3}{8\pi}\right)^{1/3} \left(\frac{hc}{4}\right) \left(\frac{\rho}{m_N}\right)^{4/3}$$

$$M \propto R^{(\gamma-2)/(3\gamma-4)}$$

$$n = 1/(\gamma - 1)$$
$$\gamma = (n+1)/n$$



Nuclear matter is not an ideal gas

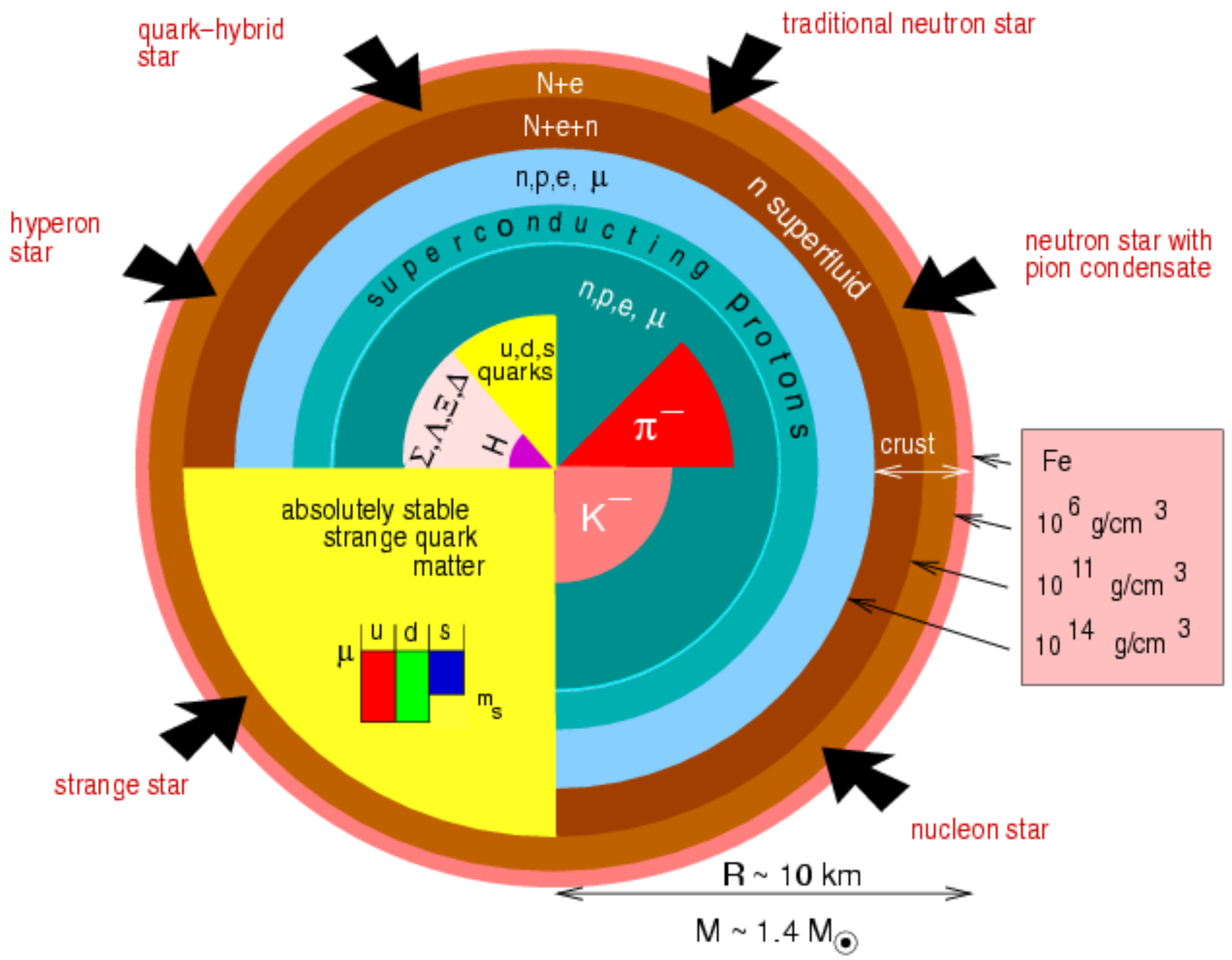
F. Weber 2005

nonrelativistic $\gamma = \frac{5}{3}$

ultrarelativistic $\gamma = \frac{4}{3}$

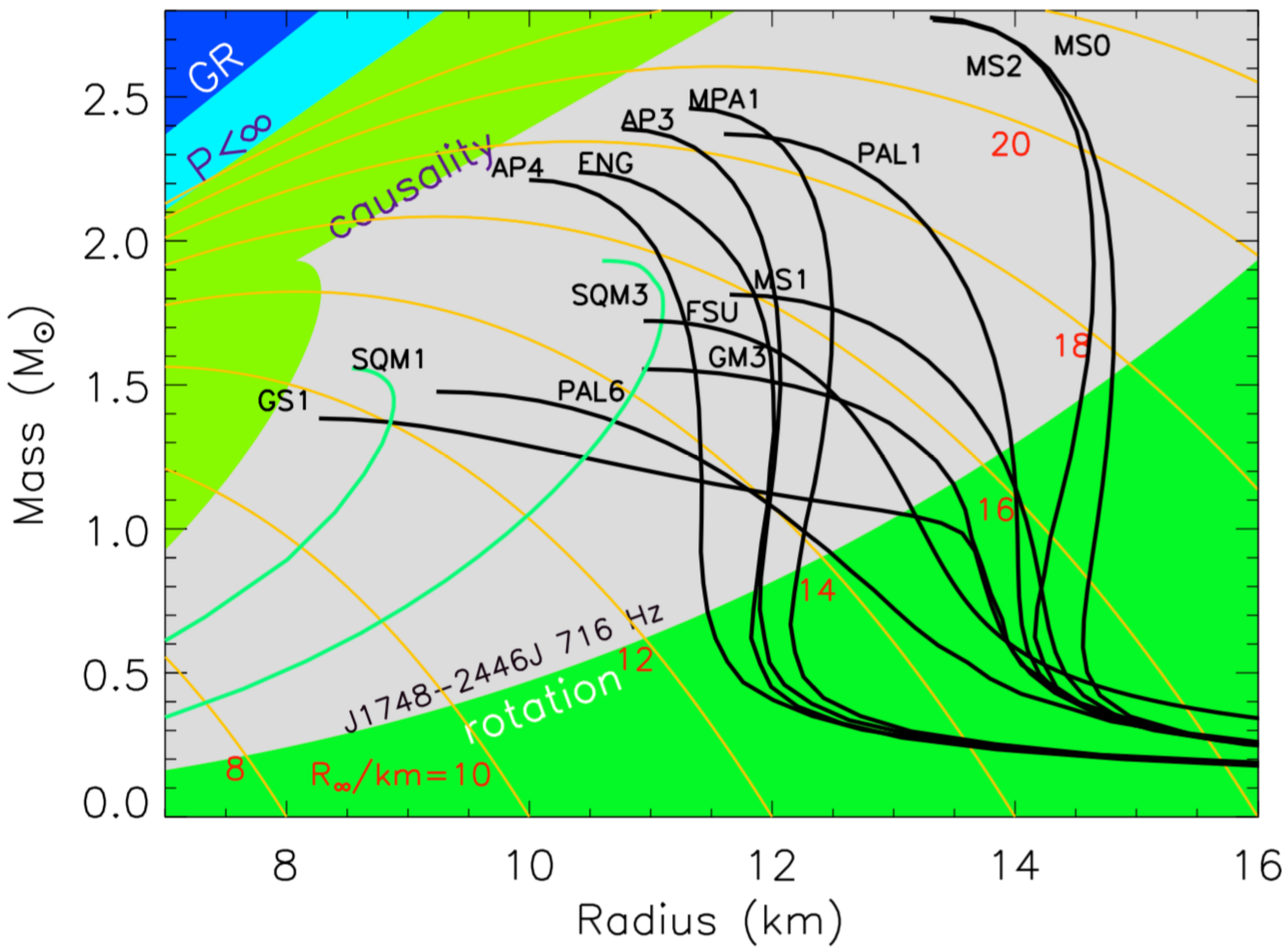
NS EOS includes

$$\gamma \approx 2 \ (n \approx 1)$$



- **still uncertain due to the nature of strong interactions**
- **introduction of 3 body forces**
- **exotic states with strangeness**
- **… …**

Neutron Star Equation of States



$$p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$$

with general relativity & strong interactions !

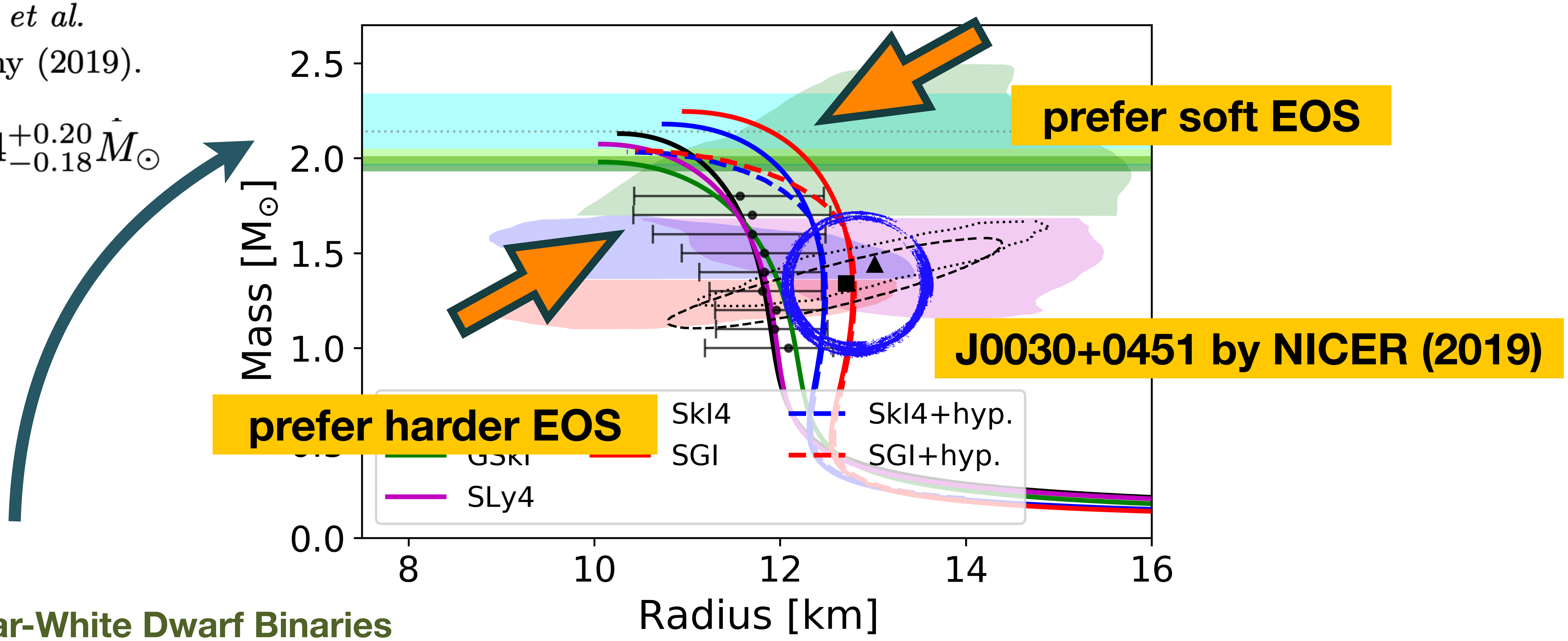
Constraints from Observations

Q) Which EOS ?

Tidal deformability of NS from GW
Strangeness in NS

H. T. Cromartie, *et al.*
Nature Astronomy (2019).

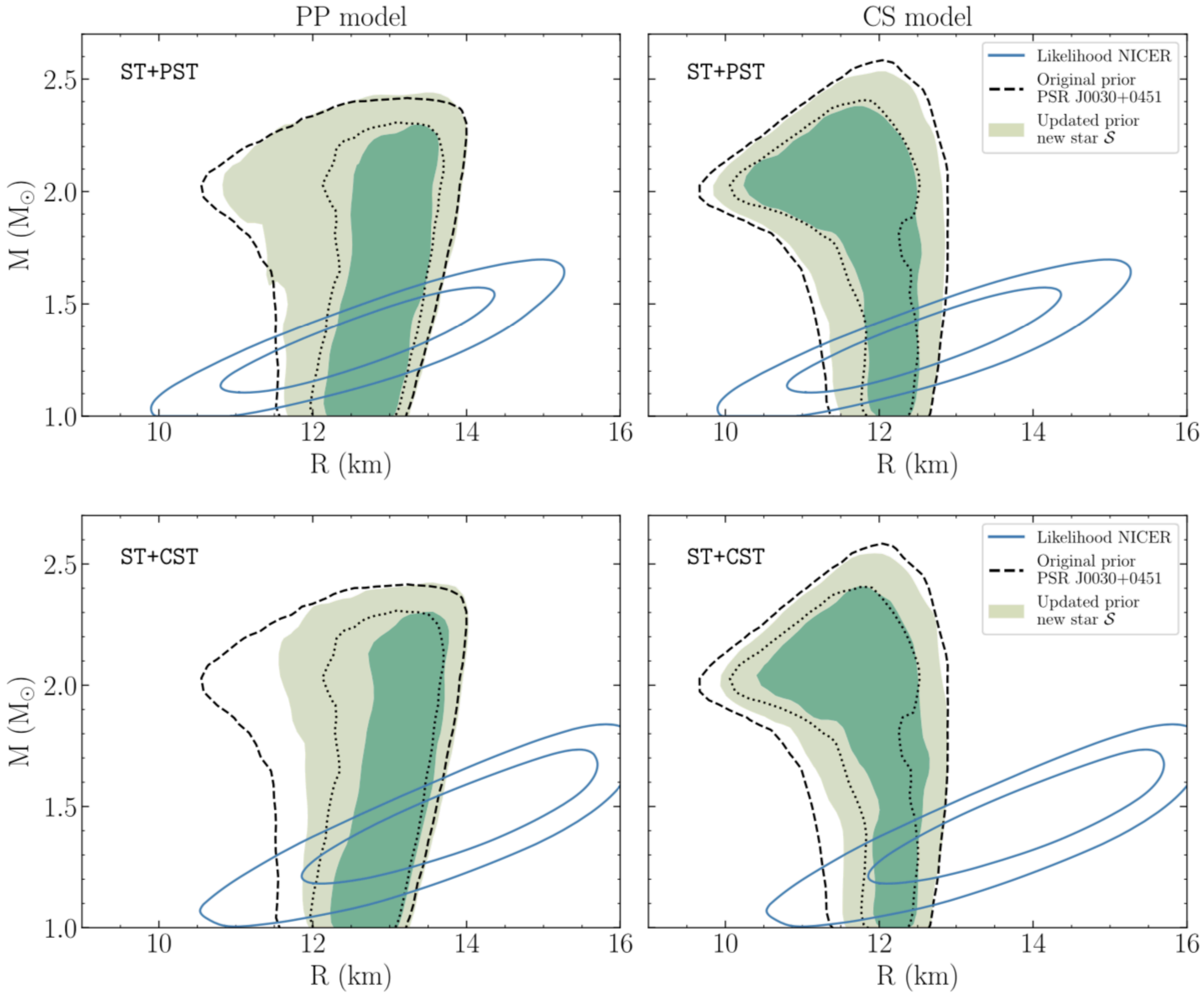
$2.14^{+0.20}_{-0.18} M_\odot$



Neutron Star-White Dwarf Binaries
1.97 solar mass NS : Nature 467 (2010) 1081
2.01 solar mass NS : Science 340 (2013) 6131

IJMPE (2020) Kim, Lee, Kim, Kwak, Lim, Hyun

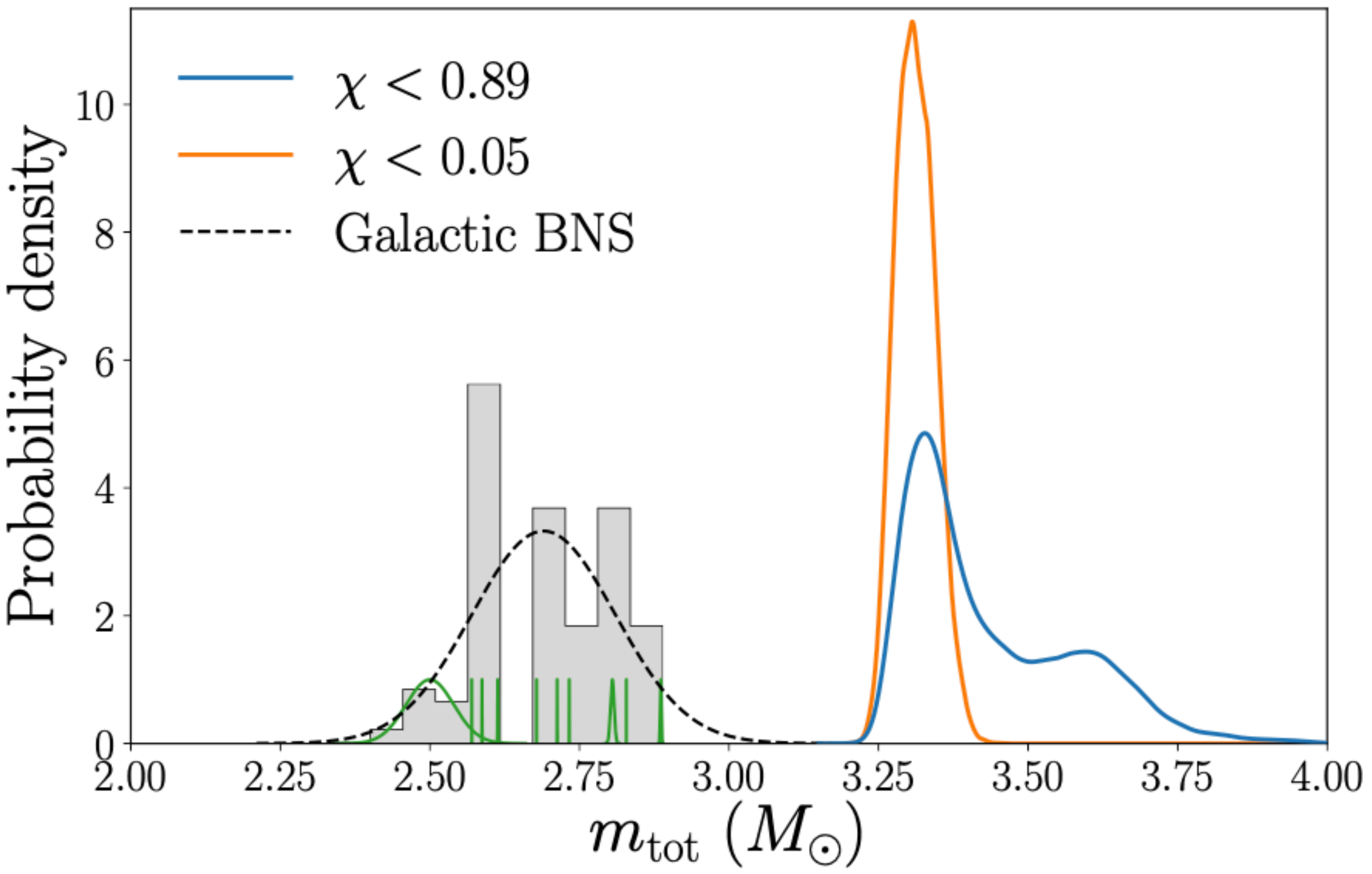
Constraints from LMXB (including PSR J0030+0451) [Raaijmakers et al. (2019)]

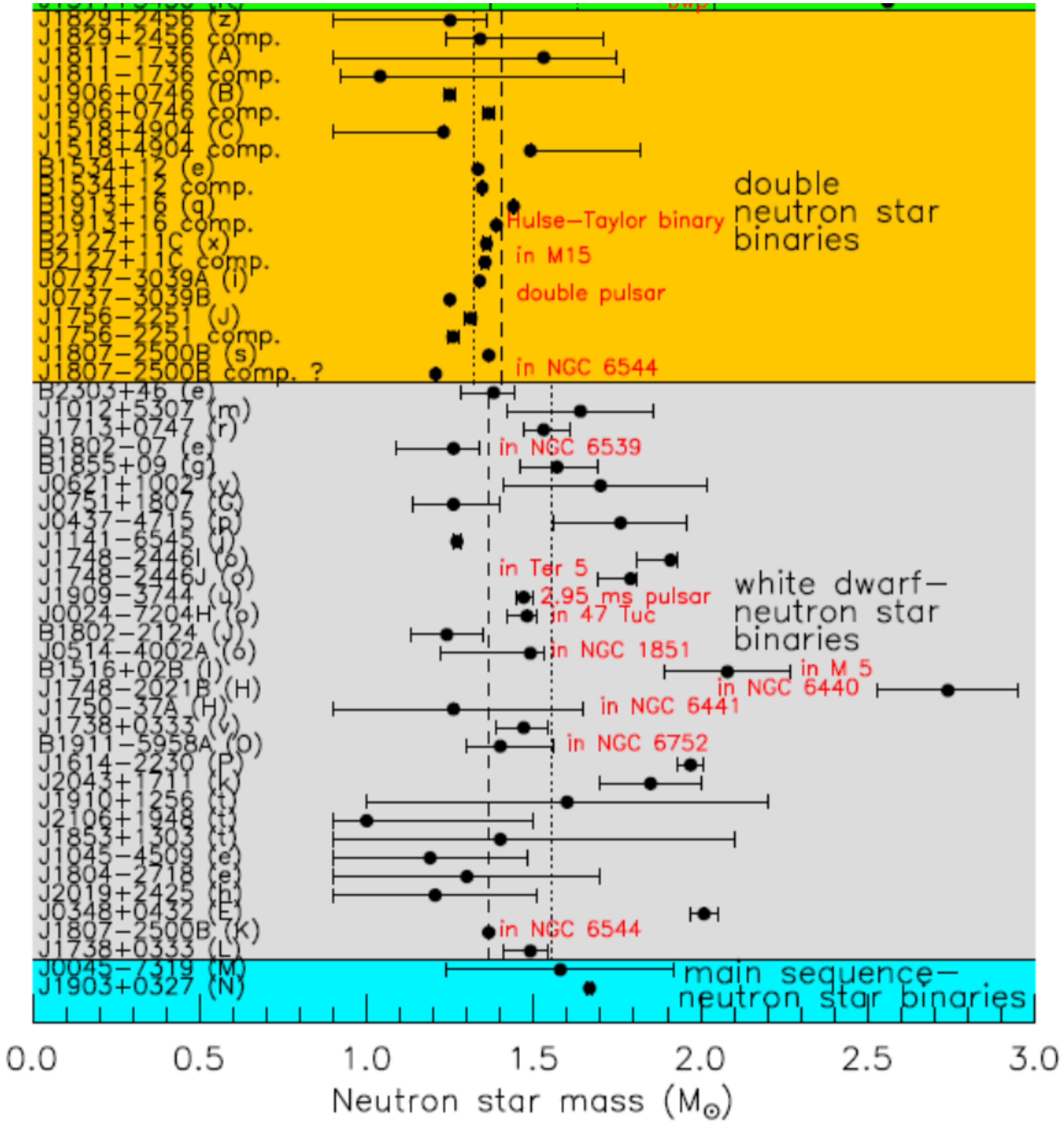


GW190425: Observation of a Compact Binary Coalescence with Total Mass $\sim 3.4\,M_\odot$

	Low-spin prior ($\chi < 0.05$)	High-spin prior ($\chi < 0.89$)
Primary mass m_1	$1.62 - 1.88\,M_\odot$	$1.61 - 2.52\,M_\odot$
Secondary mass m_2	$1.45 - 1.69\,M_\odot$	$1.12 - 1.68\,M_\odot$
Chirp mass $\mathcal{M}$	$1.44^{+0.02}_{-0.02}\,M_\odot$	$1.44^{+0.02}_{-0.02}\,M_\odot$
Detector-frame chirp mass	$1.4868^{+0.0003}_{-0.0003}\,M_\odot$	$1.4873^{+0.0008}_{-0.0006}\,M_\odot$
Mass ratio m_2/m_1	$0.8 - 1.0$	$0.4 - 1.0$
Total mass $m_{\rm tot}$	$3.3^{+0.1}_{-0.1}\,M_\odot$	$3.4^{+0.3}_{-0.1}\,M_\odot$
Effective inspiral spin parameter $\chi_{\rm eff}$	$0.013^{+0.01}_{-0.01}$	$0.058^{+0.11}_{-0.05}$
Luminosity distance $D_{\rm L}$	161^{+67}_{-73} Mpc	159^{+69}_{-71} Mpc
Combined dimensionless tidal deformability $\tilde{\Lambda}$	≤ 600	≤ 1100

GW190425: Observation of a Compact Binary Coalescence with Total Mass $\sim 3.4\,M_\odot$



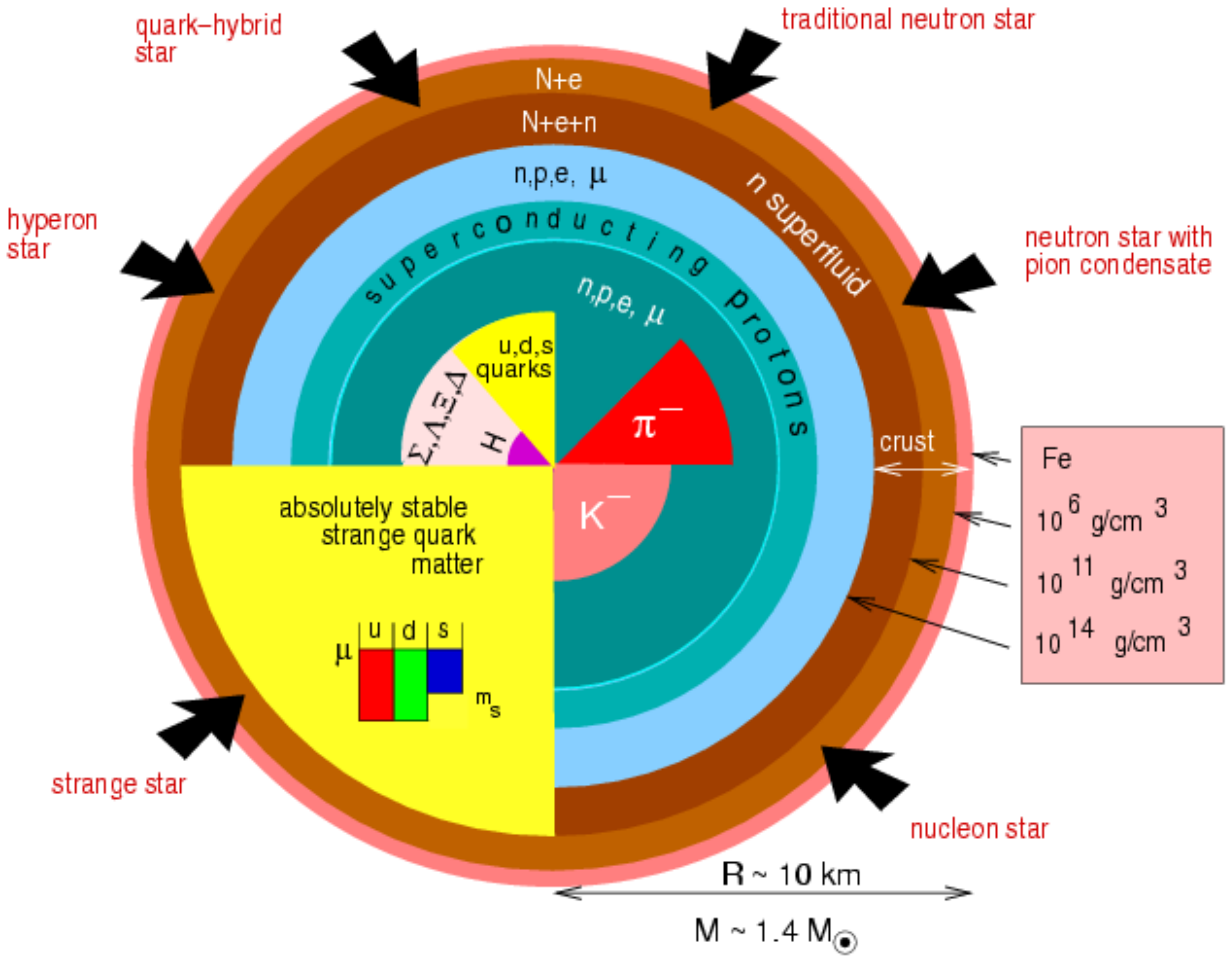


Parameterized EOS

Astrophysical Approaches

piecewise polytropic EoS (Bayesian approaches)

$$P(\rho) = K_i \rho^{\Gamma_i}$$



A new constraints by GW observations

Spectral expansion of adiabatic index [Lindblom et al.]

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

$$\Gamma(p) = \exp\left[\sum_k \gamma_k \Phi_k(p)\right]$$

$$\Gamma(x) = \exp\left(\sum_k \gamma_k\, x^k\right)$$

$$\Gamma(p) = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon}$$

$$\frac{d\epsilon(p)}{dp} = \frac{\epsilon(p) + p}{p\,\Gamma(p)}$$

$$\epsilon(p) = \frac{\epsilon_0}{\mu(p)} + \frac{1}{\mu(p)}\int_{p_0}^{p} \frac{\mu(p')}{\Gamma(p')} dp'$$

$$\mu(p) = \exp\left[-\int_{p_0}^{p} \frac{dp'}{p'\,\Gamma(p')}\right]$$

cf. piecewise polytropic EoS

$$p(\rho) = K_i \rho^{\Gamma_i}$$

[Problem 3] Any realistic neutron star equation of state cannot be represented by a single polytropic index and piecewise polytropic EOS is used in many literatures; $P(\rho) = K_i\rho^{\Gamma_i}$. In the piecewise polytropic EOS, different values of polytropic index are used for different density region. As a result, at the boundaries, discontinuities occur. Hence, instead of piecewise polytropics, in the recent analysis of tidal deformability of neutron stars [arXiv:1805.11581], spectral expansion of adiabatic index is used [arXiv:1009.0738, arXiv:1207.3744, arXiv:1807.02538];

$$\Gamma(P) = \exp\left[\sum_k \gamma_k \Phi_k(P)\right]$$

with $\Phi_k = [\ln(P/P_0)]^k$. Note that the adiabatic index is defined as

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

and $P \propto \rho^\Gamma$ when Γ is constant.

Workout Problem

a) From the first law of thermodynamics, show that

$$\Gamma(P) = \frac{\epsilon + P}{P}\frac{dP}{d\epsilon}.$$

* Number of particles is fixed.

$$d\left(\frac{\varepsilon}{\rho}\right) = -p\, d\left(\frac{1}{\rho}\right)$$

$$\frac{d\varepsilon}{\rho} - \varepsilon\frac{d\rho}{\rho^2} = P\frac{d\rho}{\rho^2}$$

$$d\varepsilon = (\varepsilon + p)\frac{d\rho}{\rho} = (\varepsilon + p)\frac{1}{\Gamma}\frac{dP}{P}$$

$$\Gamma = \left(\frac{p+\varepsilon}{p}\right)\frac{dP}{d\varepsilon}.$$

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

Workout Problem

b) Show that $\epsilon(P)$ can be obtained by

$$\epsilon(P) = \frac{1}{\mu(P)}\left(\epsilon_0 + \int_{P_0}^{P} \frac{\mu(p')}{\Gamma(p')} dp'\right)$$

$$\mu(p) = \exp\left[-\int_{P_0}^{p} \frac{dp'}{p'\Gamma(p')}\right]$$

where $\epsilon_0 = \epsilon(P_0)$ is the constant integration needed to fix the solution.

b) $$\frac{d\epsilon}{dP} = \frac{\varepsilon + P}{P\Gamma}$$

$$d\epsilon = \frac{\varepsilon+P}{P\Gamma} dP = (\varepsilon + p)\frac{dP}{P\Gamma}$$

$$\mu = \exp\left[-\int_{P_0}^{P} \frac{dP'}{P'\Gamma(P')}\right]$$

$$\frac{d\mu}{dP} = -\frac{1}{P\Gamma(P)}\mu$$

$$\frac{d\mu}{\mu} = -\frac{dP}{P\Gamma(P)}$$

$$d\epsilon = -(\varepsilon + p)\frac{d\mu}{\mu} \Rightarrow \mu d\epsilon = -(\varepsilon + p) d\mu$$

$$\Rightarrow \mu d\varepsilon + \varepsilon d\mu = -p d\mu$$

$$d(\varepsilon\mu) = d\varepsilon\mu + \varepsilon d\mu = -p d\mu$$

$$= \frac{\mu}{\Gamma} dp$$

$$\therefore \varepsilon\mu - \epsilon_0 = \int \frac{\mu}{\Gamma} dp$$

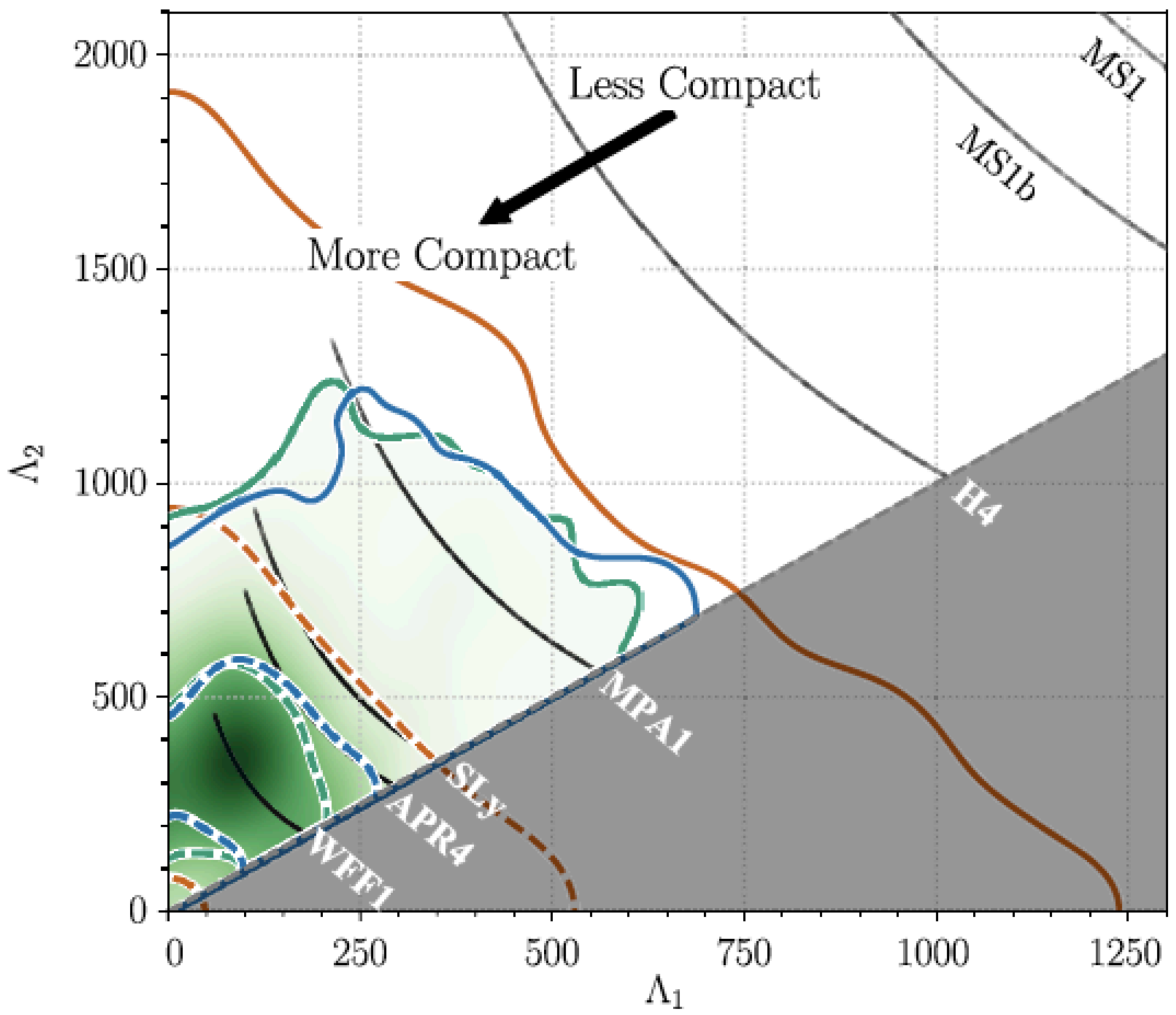
$$\epsilon\mu = \varepsilon_0 + \int \frac{\mu}{\Gamma} dP.$$

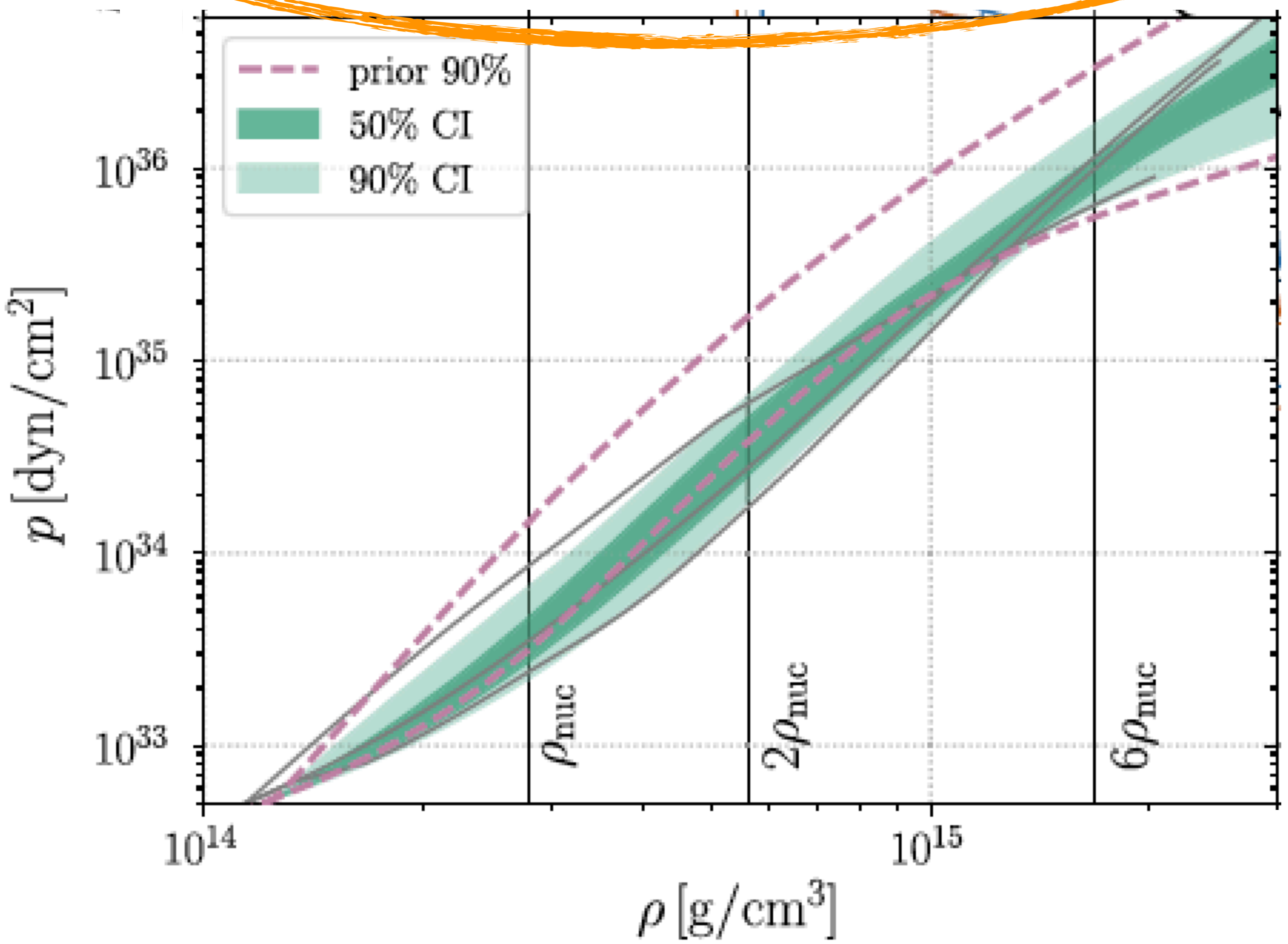
A new constraints by GW observations

$$\Lambda_{1.4M_\odot} = 190^{+390}_{-120}$$

$$P_{2\rho_{\rm nuc}} = 3.5^{+2.7}_{-1.7} \times 10^{34}\ {\rm dyne/cm^2}$$

$$P_{6\rho_{\rm nuc}} = 9.0^{+7.9}_{-2.6} \times 10^{34}\ {\rm dyne/cm^2}$$





LSC and Virgo, PRL 121, 161101 (2018)

$$\rho_{\rm nuc} = 2.8 \times 10^{14}\ {\rm g/cm^3}$$

Nature physics / Annala et al. (2020)

Evidence for quark-matter cores in massive neutron stars

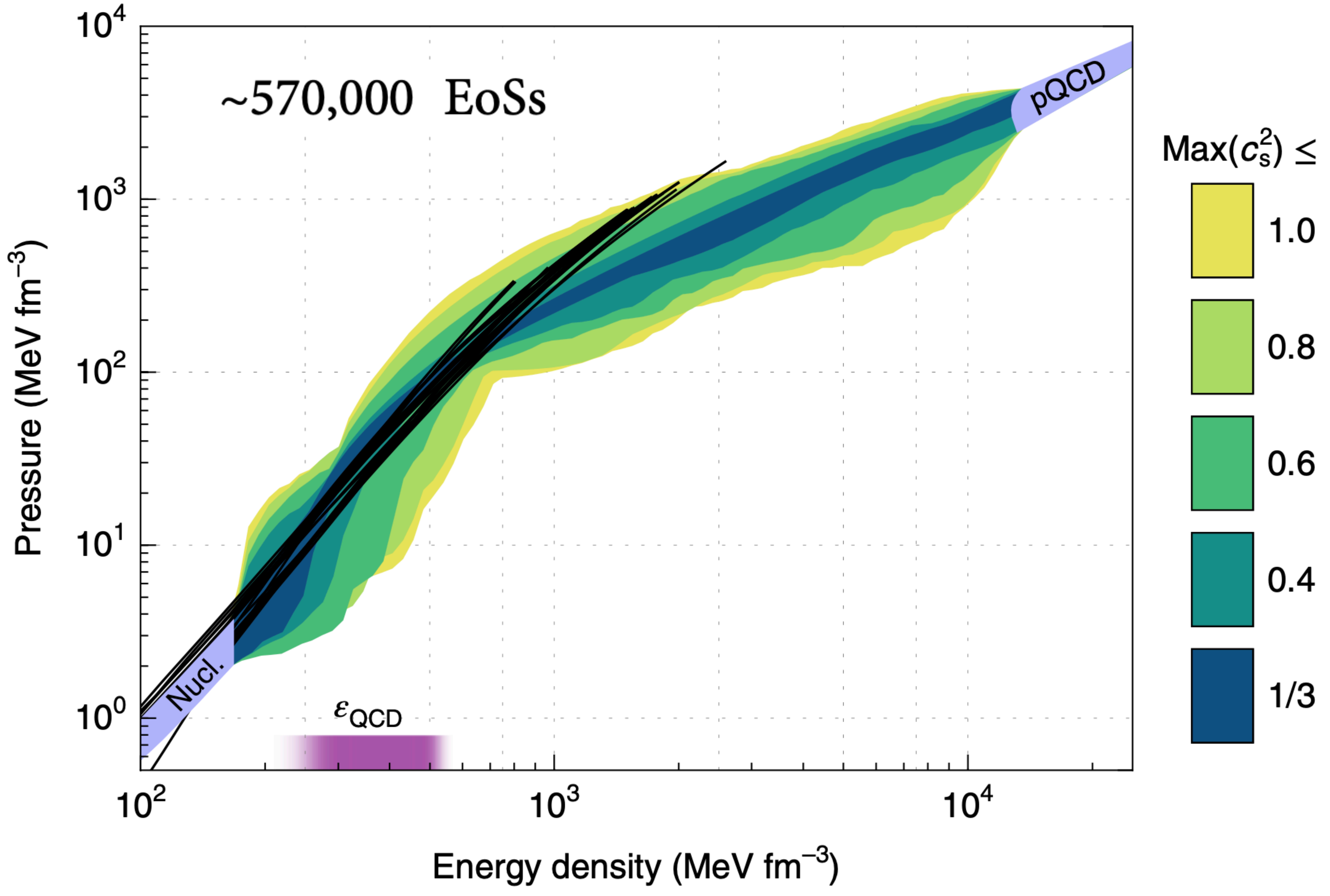
$\gamma \equiv \mathrm{d}(\ln p)/\mathrm{d}(\ln \epsilon)$

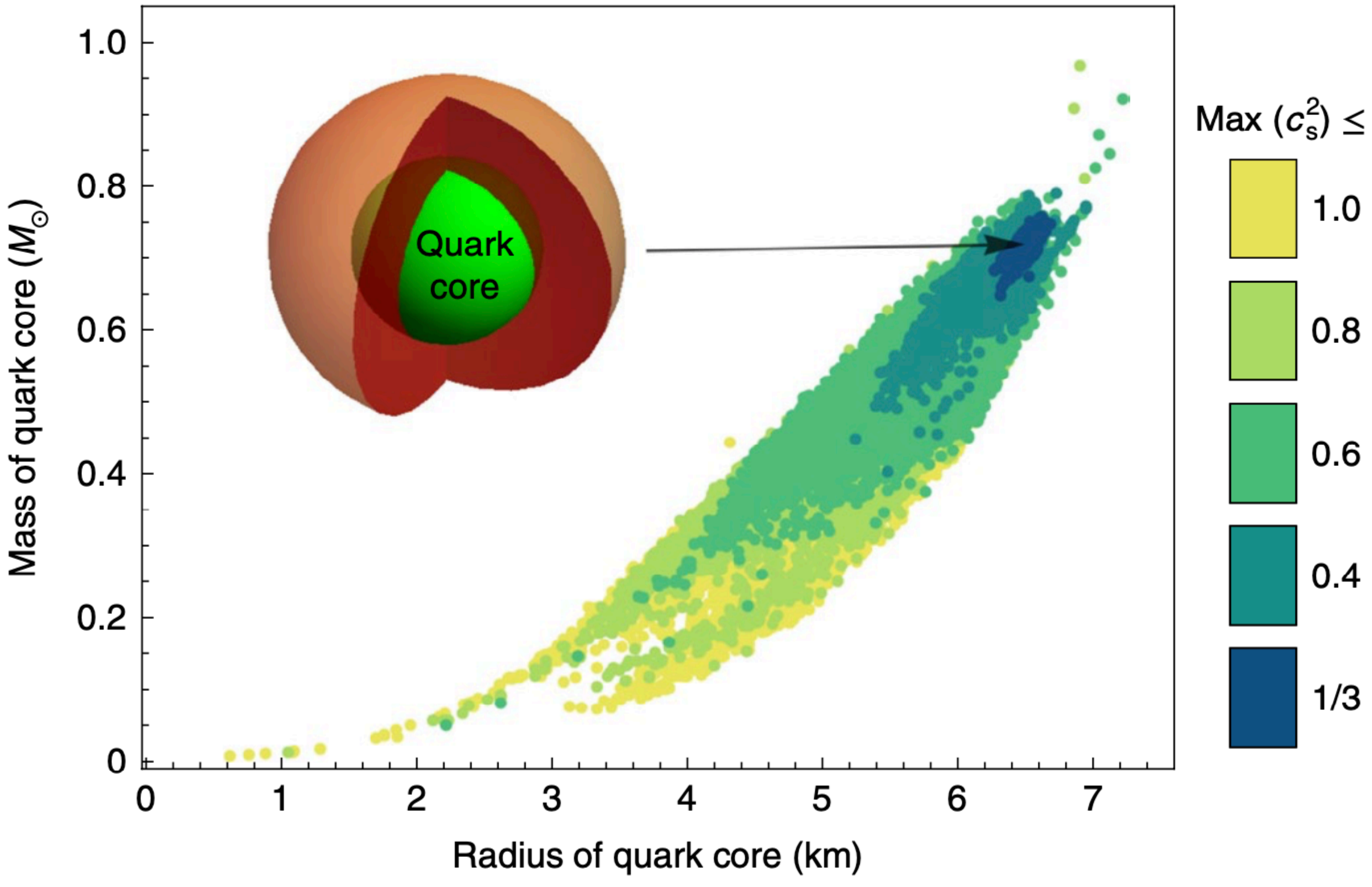
hadronic models generically predict $\gamma \approx 2.5$ around and above saturation density.

conformal matter $n \geq n_{\mathrm{pQCD}}$

$\gamma = 1$ $c_s^2 = 1/3$

$n_{\mathrm{pQCD}} = 40\ n_0$





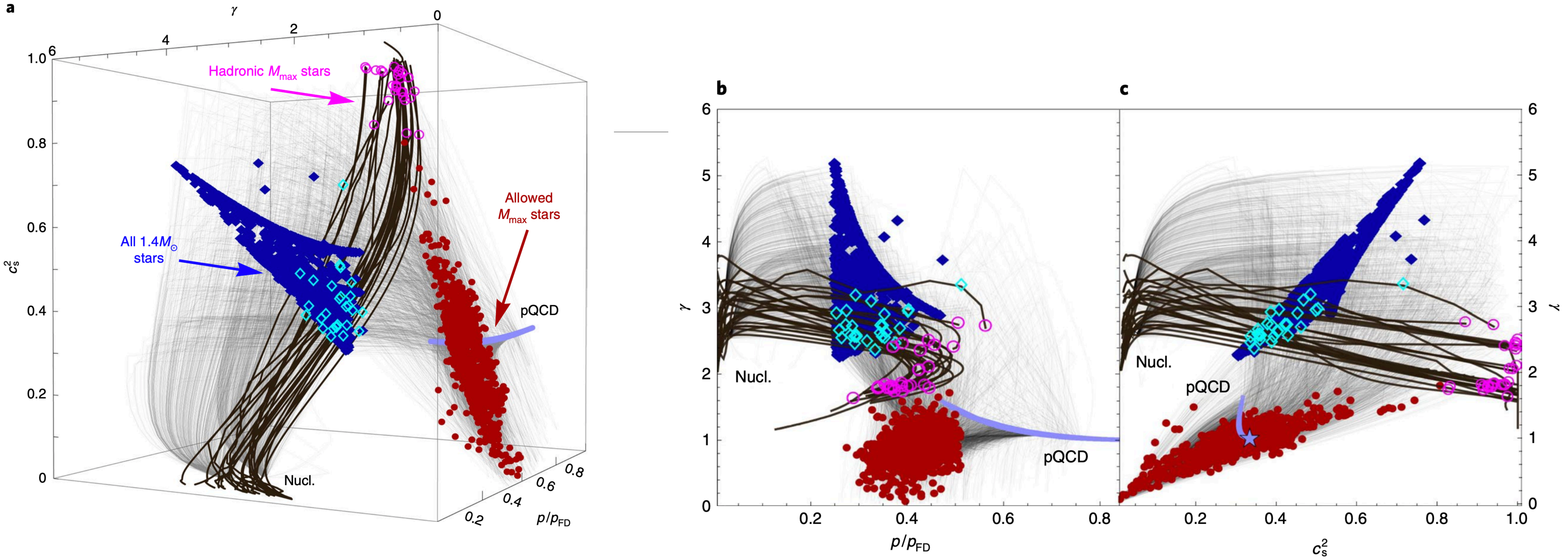


Fig. 2 | Characterization and microscopic interpretation of the equations of state. a, A 3D rendering of trajectories of a representative subset of interpolated (thin black lines) and hadronic (thick black lines) EoSs in a space spanned by the polytropic index γ, the pressure ratio p/p_{FD} and the squared speed of sound c_s^2. The solid blue and empty cyan diamonds mark the centres of $1.4M_\odot$ NSs, and the solid red and empty magenta circles denote the same for M_{max} stars. The thick light blue line denotes the region of the 3D space spanned by the high-density pQCD EoS, and the region occupied by low-density matter is indicated by the label 'Nucl'. **b**,**c**, Projection of the 3D image to the p/p_{FD}–γ (**b**) and c_s^2–γ (**c**) planes. In **c**, the light blue star indicates the high-density conformal pQCD limit. An animation of the figure is available as Supplementary Video 1.

How to understand neutron star equations of state ?

- **Massive NS from EM waves**
  - prefer hard EOS (large sound velocity) at very high densities
- **Tidal deformability & Mass distribution from Gravitational waves**
  - constraints around 2 normal matter density
- **J0030+0451 from NICER (X-ray)**
  - constraints M & R
- Nuclear & particle physics

To be continued

- One can understand the structure of neutron stars if we know the polytropic structure.
- Question is **how to determine the (density-dependent) polytropic index.**
- Nuclear & Particle Physics is essential to determine dense matter equations of state.